DETERMINANTS

FOR

HONOURS AND POST-GRADUATE STUDENTS

OF

All Indian Universities and for various Competitive Examinations

By

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PREFACE TO THE THIRD EDITION

In this edition the subject matter has been revised and a few new questions have been added.

The first chapter is followed by a miscellaneous exercise consisting of typical and important questions based on the articles given in this chapter. In the previous editions these questions were given in the second chapter. This chapter is for B. A. and B. Sc. (pass) students.

The second chapter now contains questions based upon special type of determinants, and is meant for Honours and Post-graduate students.

The university papers have been added upto date,

Any suggestion for improvement of the book will be highly appreciated.

In the end I feel it my most pleasant duty to thank one student who did not disclose his name and was good enough to send me a list of printing mistakes which occured in the last edition. The book will be free from printing mistakes as far as possible.

315, Chhipi Tank, Meerut
March '62.

M. L. KHANNA

PREFACE TO THE SIXTH EDITION

The book has been thoroughly revised and misprints removed as far as possible. The university papers have been added upto-date.

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Where is What ?

CHAPTER

ELEMENTARY PROPERTIES OF DETERMINANTS

§ 1. Consider the following two equations :—

$$a_1x+b_1y=0, \qquad \ldots(1)$$

$$a_2x+b_2y=0. \qquad \ldots(2)$$

$$\therefore \quad -\frac{a_1}{b_1}=\frac{y}{x}=-\frac{a_2}{b_2}.$$

Eliminating x and y we get $-\frac{a_1}{b_1}=-\frac{a_2}{b_2}$

or $$a_1b_2-a_2b_1=0.$$

We shall express the above eliminant in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0. \qquad \ldots(A)$$

i.e. suppress the letters to be eliminated in the given equations and enclose their coefficients as above in two parallel lines to give you the eliminant.

(A) is called a **Determinant** of second order and its value as we seen is $a_1b_2-a_2b_1$.

$$\begin{vmatrix} a_1 & b_1 \\ -a_2 & b_2 \end{vmatrix}$$

§ 2. Let us now eliminate x, y, z, from the following three equations

$$a_1x+b_1y+c_1z=0, \qquad \ldots(1)$$

$$a_2x+b_2y+c_2z=0, \qquad \ldots(2)$$

$$a_3x+b_3y+c_3z=0. \qquad \ldots(3)$$

Solving equations (2) and (3) by cross-multiplication,

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{-(a_2c_3-a_3c_2)}=\frac{z}{a_2b_3-a_3b_2}=k.$$

Substituting the values of x, y, z, in (1), we get

$k\ [a_1\ (b_2c_3-b_3c_2)-b_1\ (a_2c_3-a_3c_2)+c_1\ (a_2b_3-a_3b_2)]=0$

$\therefore$ the eliminant is

$a_1\ (b_2c_3-b_3c_2)-b_1\ (a_2c_3-a_3c_2)+c_1\ (a_2b_3-a_3b_2)=0.$

The eliminant shall be written in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0, \qquad \text{...(A)}$$

i.e. suppress the letters to be eliminated in the given equations and enclose their coefficients as above in two parallel lines to give you the eliminant.

(A) is called a determinant of 3rd order and its value as seen above is

$$a_1\ (b_2c_3-b_3c_2)-b_1\ (a_2c_3-a_3c_2)+c_1\ (a_2b_3-a_3b_2)=0$$

or $$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

§ 3. The value of a determinant is called its expansion. We find from above that a determinant of third order has *three rows* and three columns It consists of 9, *i e.* 3^2 letters which are called constituents and its expansion has 6, *i.e.* 3 ! products of the form $a_1b_2c_3$, $-b_1a_2c_3$ and so on which are called elements of the determinant.

Similarly a determinant of nth order shall have n columns n rows, n^2 constituents and n ! elements in it.

§ 4. Leading or principal term.

In determinant (A) the element $a_1 b_2 c_3$ is called the principal element or term In this element the suffixes of the constituents occur in their natural order.

Multiply the constituents diagonally starting from the left hand top corner to right hand bottom corner. The term thus obtained will be the principal term. The above determinant is sometimes expressed as $(a_1 b_2 c_3)$ *i.e.* by enclosing within brackets its principal term.

§ 5. Expansion of a determinont—We have seen that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a\,(b_2c_3 - b_3c_2) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2) \quad . \text{(A)}$$

or $$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The above is called expansion with 1st row.

or a_1 (determinant obtained by removing the row and column intersecting at a_1)

$-b_1$ (determinant obtained by removing the row and column intersecting at b_1)

$+c_1$ (determinant obtained by removing the row and column intersecting at c_1).

Now by rearranging the terms of (A), we find that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\,(b_2c_3 - b_3c_2) - a_2\,(b_1c_3 - b_3c_1) + a_3\,(b_1c_2 - b_2c_1)$$

which is called expansion with 1st column, The expansion with 1st column is obtained in the same way as the expansion with 1st row explained before.

Similarly if we have a determinant of fourth order.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

then following the same law of expansion we can write it as with 1st row,

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

or expanding with 1st column the above determinant is equal to

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Note. The above rule of expansion is true when the determinant is expanded with either first row or first column. We will always expand the determinant with either 1st row or 1st column. Expansion can however be made with any other row or column provided such row of column is made the 1st row or column as will be explained further in Art 9.

*** § 6. The value of a determinant is not altered by changing the rows into columns, and the columns into rows. (Proof)** **(Cal. Pass 65)**

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \ldots (A)$

Let a new determinant Δ' be formed by changing the rows into columns and columns into rows of Δ.

$$\text{Let } \Delta=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=a_1\,(b_2c_3-b_3c_2)-a_2\,(b_1c_3-b_3c_1)+a_3\,(b_1c_2-b_2c_1), \quad \ldots(B)$$

Now after re-arranging the terms of (B) it can be put as

$$a_1\,(b_2c_3-b_3c_2)-b_1\,(a_2c_3-a_3c_2)+c_1\,(a_2b_3-a_3b_2),$$

which is same as A. $\therefore$ $\Delta'=\Delta$

§ 7. (a) **If two adjacent rows or columns of a determinant are interchanged, the sign of the determinant is changed, whereas its numerical value remains the same. (Proof)**

(Agra B Sc. 58)

$$\text{Let } \Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}=a_1\,(b_2c_3-b_3c_2)-b_1\,(a_2c_3-a_3c_2)+c_1\,(a_2b_3-a_3b_2). \quad \ldots(A)$$

Let a new determinant Δ be formed by interchanging the 1st and 2nd rows of Δ.

$$\therefore \quad \Delta'=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_2\,(b_1c_3-b_3c_1)-b_2\,(a_1c_3-a_3c_1)+c_2\,(a_1b_3-a_3b_1). \quad \ldots(B)$$

Rearranging the terms of (B), it can be put in the form

$$-\{a_1\,(b_2c_3-b_3c_2)-b_1\,(a_2c_3-a_3c_2)+c_1\,(a_2b_3-a_3b_2)\}$$

which is same as A except being of opposite sign.

$$\therefore \quad \Delta'=-\Delta.$$

(b) If any two rows or two columns of a determinant are interchanged, the determinant retains its absolute valne but changes sign.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$. ...(A)

Let a new determinant Δ' be formed by interchanging the first and third rows.

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2). \quad ...(B)$$

Re-arranging the terms of (B) it can be put in the form $-\{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$ which is same as (A) except being of opposite sign. $\therefore\ \Delta' = -\Delta$.

§ 8. If any line of a determinant be passed over three parallel lines, then the resulting determinant will be $(-1)^3$ of the original determinant.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

Let a new determinant Δ' be formed by making the fourth row of Δ the 1st row of Δ' *i.e.* by making it cross over three parallel lines; then we shall prove that

$$\Delta' = (-1)^3\,\Delta.$$

$$\Delta' = \begin{vmatrix} a_4 & b_4 & c_4 & d_4 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

make the third column take the position of 1st column for which we shall have to pass it over 2 columns and hence by article 8, $\Delta=(-1)^2 \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$

$$=(-1)^2 [c_1 (a_2b_3-a_3b_2)-c_2 (a_1b_3-a_3b_1)+c_3 (a_1b_2-a_2b_1)]$$
$$=(-1)^2 [a_1 (b_2c_3-b_3c_2)-b_1 (a_2c_3-a_3c_2)+c_1 (a_2b_3-a_3b_2)]$$
$$=(-1)^2 \Delta.$$

Similarly expansion with 2nd column will be equal to $(-1)^1 \Delta$.

Hence we need not undergo the procedure to bring the row or column with which we want to expand, in the position of first row or column. We may expand with any row or column in accordance with the rules of article 5 but multiply with $(-1)^p$ where p is the number of crossings of lines which will make the expanding row or column 1st row or column.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ by expanding with 3rd row

$$=(-1)^2 [a_3 (b_1c_2-b_2c_1)-b_3 (a_1c_2-a_2c_1)+c_3 (a_1b_2-a_2b_1)]$$

where 2 is the number of crossings of lines which will make the third row take the position of the 1st row.

Above can also be written as

$$(-1)^1 [b_1 (a_2c_3-a_3c_2)-b_2 (a_1c_3-a_3c_1)+b_3 (a_1c_2-a_2c_1)]$$

***§ 10. If any two rows or two columns of a determinant are identical the determinant vanishes.**

(Delhi 53 ; Pb. 35, 39 ; Sagar 66)

Let a determinant Δ be such in which 2nd and 3rd columns are identical.

$$\Delta = \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

Let a new determinant, Δ' be formed by interchanging the identical columns.

$$\Delta' = \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

The value of Δ' by article no. 7, must be equal to $-\Delta$. But actually we find that there is no differnce between Δ' and Δ ; $\Delta' = \Delta$ but $\Delta' = -\Delta$ (Art. 7).

Hence $\Delta = -\Delta$ or $2\Delta = 0$; $\therefore$ $\Delta = 0$.

§ 11. Minor.

The determinant that is left by cancelling the row and column intersecting at a particular constituent is called the minor of that constituent and is denoted by the corresponding capital letter.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Minor of } b_2 = B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

Minor of $c_3 = C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\text{or } a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Both the above forms can be put as

$$a_1 A_1 - b_1 B_1 + c_1 C_1$$

or $$a_1 A_1 - a_2 A_2 + a_3 A_3,$$

where capital letters denote the minors of the correspondin small letters.

If expanded with 2nd row or 3rd row.

$$\Delta = (-1)^1 \left\{ a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \right\}$$

$$\text{or } (-1)^2 \left\{ a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\}$$

Both these can be put as

$$(-1)^1 (a_2 A_2 - b_2 B_2 + c_2 C_2)$$

or $$(-1)^2 (a_3 A_3 - b_3 B_3 + c_3 C_3),$$

$$\therefore \quad a_2 A_2 - b_2 B_2 + c_2 C_2 = -\Delta,$$
$$a_3 A_3 - b_3 B_3 + c_3 C_3 = \Delta,$$
$$a_1 A_1 - b_1 B_1 + c_1 C_1 = \Delta,$$
$$a_1 A_1 - a_2 A_2 + a_3 A_3 = \Delta.$$

Hence when the constituents of any row or column are multiplied with their corresponding minors and summed up with alternatively plus and minus, signs, the result $= \Delta$ or $-\Delta$

according as the row or column whose minors are written is made the first row or column by even or odd crossings of rows or columns.

Now $a_1A_1-b_1B_1+c_1C_1=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\Delta.$

$\therefore\quad a_2A_1-b_2B_1+c_2C_1=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$ (Art. 10) Two rows identical

Similarly

$a_1A_1-a_2A_2+a_3A_3=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\Delta.$

But $b_1A_1-b_2A_2+b_3A_3=\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}=0$ (Art. 10) Two colums identical

Hence if the constituents of any row or column are multiplied with the minors of the constituents of some other row or column and summed up with alternately plus and minus sign, the result $=0$. (Gauhati 65 Hons.)

$$\therefore\quad b_1C_1-b_2C_2+b_3C_3=0$$

$$a_3A_1-b_3B_1+c_3C_1=0.$$

§ 12. Co-factor.

The determinant that is left by cancelling the row and column intersecting at a particular constituent, when that

particular constituent is brought in the top left hand corner of a determinant is called its co-factor and is denoted by corresponding capital letter.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Co-factor of c_2 : Now c_2 can be brought in top left hand corner by one movement of row and two movements of columns *i e.* in all 3 movements

$$C_2 = \text{co-factor of } c_2 = (-1)^3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -(\text{minor of } c_2)$$

$$B_2 = \text{co-factor of } b_2 = (-1)^2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = \text{minor of } b_2.$$

Hence the co-factors are equal to minors in magnitude but are of same or of opposite sign according as the letters whose co-factor is being calculated is brought to top left hand corner by even or odd movements of lines :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3$$

in terms of minors

$$= a_1A_1 + a_2A_2 + a_3A_3$$

in terms of co-factors.

Just like in article No. 11, it is quite clear that

$$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$

$$b_1B_1+b_2B_2+b_3B_3=\Delta, \text{ (Note)}$$
$$a_3A_3+b_3B_3+c_3C_3=\Delta,$$
$$a_2A_2+b_2B_2+c_2C_2=\Delta \text{ (Note)}$$
$$a_1A_2+b_1B_2+c_1C_2=0,$$
$$a_3A_1+b_3B_1+c_3C_1=0,$$

where capitial letters denote the co-factors of corresponding small letters.

The expansion of a determinant in terms of minors and co-factors differ only in that in the former the terms are alternately plus and minus while in the latter all the terms are plus and that if the determinant be expanded with any row or any column the result is either zero or $+\Delta$ (and not $-\Delta$ as in minor).

§ 13. If each constituent in any row or in any column be multiplied by the same factor, then the determinant is multiplied by that factor.

Let $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1A_1-b_1B_1+c_1C_1.$

Let a new determinant Δ' be formed by multiplying the constituents of 1st row by a constant m

$$\Delta'=\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Now the minors of ma_1, mb_1 and mc_1 in Δ' are the same as those of a_1, b_1 and c_1 in Δ.

$$\therefore \quad \Delta'=ma_1A_1-mb_1B_1+mc_1C_1.$$

$$=m\,(a_1A_1-b_1B_1+c_1C_1)=m\Delta,$$

which proves the proposition.

Note 1. From above it is quite clear that $\Delta=\dfrac{1}{m}\,\Delta'$.

Hence if in a given determinant Δ the constituents of any row or column be multiplied by a constant m and a determinant Δ' be formed, then

$$\Delta=\frac{1}{m}\,\Delta'.$$

Note 2. From above it is quite clear that if in a determinant the constituents of any row or any column are k times the corresponding constituents of any other row or column, then the determinant vanishes,

i.e.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix}=k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}=0$$

$\because$ two rows are identical.

***§ 14. If each constituent of any row or column consists of two terms, then the determinant can be expressed as the sum of two determinants.** **(Proof)**

If A_1, A_2 and A_3 are the minors of a_1, a_2 and a_3 resp. in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \text{then}\ \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$$

$$=(a_1+\alpha_1)\,A_1-(a_2+\alpha_2)\,A_2+(a_3+\alpha_3)\,A_3$$

$$=(a_1A_1-a_2A_2+a_3A_3)+(\alpha_1A_1-\alpha_2A_2+\alpha_3A_3)$$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Similarly
$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & b_1+\beta_1 & c_1 \\ a_2 & b_2+\beta_2 & c_2 \\ a_3 & b_3+\beta_3 & c_3 \end{vmatrix}+\begin{vmatrix} \alpha_1 & b_1+\beta_1 & c_1 \\ \alpha_2 & b_2+\beta_2 & c_2 \\ \alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix}$$

$$+\begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

i e. sum of four determinants.

We can generalise above as follows.

If the constituent of the three column consist of [illegible] terms resp., the determinant can be expressed [illegible] *m n p* determinants.

***§ 15. If to each constituent of [illegible] added or subtracted the equi-multiples of the [illegible]**

constituents of one or more lines, the determinant remains unaltered.

(Nagpur 61 ; Mysore 59 ; Punjab 34, 37, 44, 46, 51 ; Agra 60 ; Delhi 57)

Let $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let another determinant Δ' be formed by adding p times and subtracting q times the constituents of 2nd and 3rd columns respectively from the corresponding constituents of the first column.

$$\therefore \quad \Delta'=\begin{vmatrix} a_1+pb_1-qc_1 & b_1 & c_1 \\ a_2+pb_2-qc_2 & b_2 & c_2 \\ a_3+pb_3-qc_3 & b_3 & c_3 \end{vmatrix}$$

By § 14 it can be broken into sum of three determinants.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix}$$

$$-\begin{vmatrix} qc_1 & b_1 & c_1 \\ qc_2 & b_2 & c_2 \\ qc_3 & b_3 & c_3 \end{vmatrix}=\Delta$$

Now by § 13, the last two determinants vanish.

$$\therefore \quad \Delta'=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In practice generally we can alter several lines of the determinant without altering its value.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1+sc_1 & c_1 \\ a_2+pb_2+qc_2 & b_2+sc_2 & c_2 \\ a_3+pb_3+qc_3 & b_3+sc_3 & c_3 \end{vmatrix}$$

We can split the determinant in the R. H. S. into 3×2 determinants all of which shall vanish except

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

§ 16. If any rth line has been altered by means of sth and tth parallel lines, then sth and tth cannot themselves be altered by means of that rth line, but we can use the rth line to alter all other parallel lines except rth and sth.

If
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

let
$$\Delta' = \begin{vmatrix} a_1+pb_1+qc_1 & b_1+ra_1+sc_1 & c_1 \\ a_2+pb_2+qc_2 & b_2+ra_2+sc_2 & c_2 \\ a_3+pb_3+qc_3 & b_3+ra_3+sc_3 & c_3 \end{vmatrix}$$

Δ' can be splitted into $3\times 3=9$ determinants. Out of these nine, only seven determinants shall be easily seen to be vanishing and the remaining two will be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & ra_1 & c_1 \\ pb_2 & ra_2 & c_2 \\ pb_3 & ra_3 & c_3 \end{vmatrix}$$

$$= \Delta - pr\,\Delta = \Delta\,(1-pr).$$

$\therefore\ \Delta \neq \Delta'$ because in altering the first column, the second column has benn used and again contrary to our article above in altering the second column, the first has been used and hence $\Delta \neq \Delta'$.

But $\Delta' = \begin{vmatrix} a_1+pc_1 & b_1+qa_1 & c_1 \\ a_2+pc_2 & b_2+qa_2 & c_2 \\ a_3+pc_3 & b_3+qa_3 & c_3 \end{vmatrix}$.

Δ' can be broken into 2×2 *i.e.* 4 determinants all of which will vanish except.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore\ \Delta' = \Delta.$$

Here the 2nd column has not been used in altering the first column, though the 2nd has been altered by means of first.

§ 17. If those constituents of a determinant, which involve x, are polynomials in x and if $\Delta = 0$ when 'a' be substituted for x, then x—a is a factor of the determinant.

Now since the constituents of Δ involving x are polynomials in x, then the expansion of Δ will also be a polynomial in x and hence let Δ, when expanded, give

$$\Delta=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_{n-1}x+p_n.$$

But $0=p_0a^n+p_1a^{n-1}+p_2a^{n-2}+\dots p_{n-1}a+p_n$ (given).

Substracting,

$$\Delta=p_0(x^n-a^n)+p_1(x^{n-1}-a^{n-1})+p_2(x^{n-2}-a^{n-2})+\dots +p_{n-1}(x-a)$$

which shows that $x-a$ is a factor of Δ.

Similarly if the constituents of a determinant are in terms of a, b and c and by putting $a=b$, the determinant vanishes (*i.e.* any of the two parallel lines become identical) then $(a-b)$ will be a factor of the determinant.

***Example 1.** Factorize

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

(Nagpur 30)

1st Method.

Subtracting the constituents of 2nd row from first row, and the constituents of third row from first row, *i e.* R_1-R_2 and R_2-R_3, the determinant remains unchanged by § 15,

$$\therefore\ \Delta=\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$ Now take $(a-b)$ from first row and $(b-c)$ from 2nd row (§ 13).

$$\therefore\ \Delta=(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$ Expanding with first column.

$$=(a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}=(a-b)(b-c)(b+c-a-b)$$
$$=(a-b)(b-c)(c-a)$$

2nd Method.

Here if we put $a=b$, then the first and second rows

become identical and hence $(a-b)$ is a factor. Similarly it is easily seen that $(b-c)$, $(c-a)$ are also its factors.

$\therefore$ $(a-b)(b-c)(c-a)$ is a factor of the determinant. Now the determinant is of 3rd degree as is judged from diagonal term, and the factors that we have found are also of 3rd degree. Hence there can be no other factor in terms of a, b and c. But a constant can be a factor.

$$\therefore \quad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = K(a-b)(b-c)(c-a). \qquad \text{...(A)}$$

Now in order to find k, we give a, b and c such small values so that on putting them in the determinant its expansion becomes easier. *But care is to be taken in* choosing the values, lest the determinant may become zero and the expression on the right also vanishes.

Let $a=0$, $b=1$, $c=-1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = K(0-1)(1+1)(-1-0)$$

$$2=2K; \quad \therefore \quad K=1.$$

Hence
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

The set of values, $a=0$, $b=1$, $c=1$ is not applicable because both L. H. S. as well as R. H. S. vanish.

Note. Above is the general method of finding the constant factor. The constant factor in the above example could however be found by comparing the co-efficients

of bc^2 (diagonal term) in both sides of Δ, and you will find that $K=1$.

Note. Now see just now Q. 2, 3 and 4 of Miscellaneous Exercise at the end of the chapter Page 69–72.

***Example 2. Factorize.**

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$ **(Delhi 54 : Bombay 58 ; Agra 46 ; Pb. 50)**

and find its value when $a+b+c=0$.

1st Method.

Proceeding exactly as in Ex. 1, by c_1-c_2, c_2-c_3,

$$\Delta=\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$\Delta=(a-b)\,(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$ Expanding with 1st row.

$$\Delta=(a-b)\,(b-c)\begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

$$=(a-b)\,(b-c)\,(b^2+bc+c^2-a^2-ab-b^2)$$
$$=(a-b)\,(b-c)\,[(c-a)\,(c+a)+b\,(c-a)]$$
$$=(a-b)\,(b-c)\,(c-a)\,(a+b+c)$$

2nd Method.

Just as in Ex. 1, $(a-b)$, $(b-c)$, $(c-a)$ is seen to be a factor of above. Now [illegible] diagonal term and hence we have to find one more factor of first degree in terms of a, b, c [illegible] factors found are of 3rd degree. [illegible]

The results of Ex. [illegible]

factors found are cyclic. $\therefore$ the remaining factors must also be cyclic and of first degree. But a cyclic factor of first degree in terms of a, b and c can only be $a+b+c$.

$$\therefore \Delta=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=K(a-b)(b-c)(c-a)(a+b+c)$$

Now the set of values $a=0$, $b=1$, $c=-1$ given in previous question is not applicable because R.H.S vanishes by substituting these values.

Choosing another set of values as $a=0$, $b=1$, $c=2$.

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{vmatrix}=K(0-1)(1-2)(2-0)(0+1+2).$$

$$6=6K; \quad \therefore \quad K=1.$$

Hence $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c).$

Note. In case $a+b+c=0$, then clearly $\Delta=0$.

***18. Multiplication of determinants.**

(Nagpur 61 ; Venkateswara 59 ; Agra 36, 45, 60 ; Pb. 40, 43, 46, 60)

Let $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and $\Delta'=\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$.

Then $\Delta\Delta'=D$ where D is the determinant below.

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}.$$

Now by § 14 the above determinant can be expressed as the sum of 27 determinants, a few of which will be

$$\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 \end{vmatrix}, \quad \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix} \quad \begin{vmatrix} a_1\alpha_1 & c_1\gamma_2 & b_1\beta_3 \\ a_2\alpha_1 & c_2\gamma_2 & b_2\beta_3 \\ a_3\alpha_1 & c_3\gamma_2 & b_3\beta_3 \end{vmatrix}$$

(1) (2) (3)

$$(1) \;=\alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix} = 0 \quad \text{(Identical lines).}$$

$$(2) \;=\alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$(3) \;=\alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = -\alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

[on interchanging 2nd and 3rd columns] $=-\alpha_1\gamma_2\beta_3\Delta$.

Similarly it can be seen that out of 27 component determinants D, 21 vanish whereas the rest 6 can be written as

$\alpha_1\beta_2\gamma_3\Delta$, $-\alpha_1\beta_3\gamma_2\Delta$, $-\alpha_2\beta_1\gamma_3\Delta$, $\alpha_2\beta_3\gamma_1\Delta$,
$\alpha_3\beta_1\gamma_2\Delta$, $-\alpha_3\beta_2\gamma_1\Delta$.

Summing up these six determinants, we get

$$D=\Delta\;[\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)-\alpha_2(\beta_1\gamma_3-\beta_3\gamma_1)+\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)]$$

or $$D=\Delta \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \Delta\times\Delta'.$$

$\therefore$ In the same way we can multiply the determinants any order. *Another proof is given in 2nd chapter.*

By multiplying a row of Δ with a row of Δ', we mean to multiply the corresponding constituents of two rows and adding them *i.e.* multiplying 1st row of Δ with 3rd row of Δ'.

$$\begin{matrix} a_1 & b_1 & c_1, \\ a_3 & \beta_3 & \gamma_3 \end{matrix} \quad \text{we get} \quad a_1a_3+b_1\beta_3+c_1\gamma_3 \quad \text{or} \quad \Sigma a_1a_3.$$

Hence our rule of multiplication is as follows :

Take the first row of Δ and multiply it successively with 1st, 2nd and 3rd rows of Δ' as explained above. The expressions thus obtained will be constituents of the 1st row (in order) of (D).

Then take up second row of Δ and multiply it successively with 1st, 2nd, 3rd rows of Δ' as explained above. The three expressions thus obtained will be constituents of 2nd row (in order) of (D).

In a similar manner 3rd row of (D) is obtained.

***Ex. 4. Find the value of**

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$$

(Agra 27, 47 Supp.

In accordance with the rule in the previous article, the constituents of the first row of the new determinant will be

$$\text{1st constituent}=\lambda\ (a^2+\lambda^2)+c\ (ab+c\lambda)-b\ (ca-b\lambda)$$
$$=\lambda\ (a^2+b^2+c^2+\lambda^2).$$

$$\text{2nd constituent}=-c\ (a^2+\lambda^2)+\lambda\ (ab+c\lambda)+a\ (ca-b\lambda)=0$$
$$\text{3rd constituent}=b\ (a^2+\lambda^2)-a\ (ab+c\lambda)+\lambda\ (ca-b\lambda)=0.$$

Similarly find the constituents of 2nd and 3rd rows.

$$\therefore\quad D=\begin{vmatrix} \lambda\ (a^2+b^2+c^2+\lambda^2) & 0 & 0 \\ 0 & \lambda\ (a^2+b^2+c^2+\lambda^2) & 0 \\ 0 & 0 & \lambda\ (a^2+b^2+c^2+\lambda^2) \end{vmatrix}$$

Eliminating y and z from above we get

$$\begin{vmatrix} b_1 & c_1 & a_1x-d_1 \\ b_2 & c_2 & a_2x-d_2 \\ b_3 & c_3 & a_3x-d_3 \end{vmatrix}=0$$

or $$x\begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}=\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

$$\therefore\ x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}\div\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly $$y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}\div\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}\div\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

§ 20. Notation.

We shall in the questions denote columns by C and rows by R. Thus C_1, C_2, C_3, stand for 1st, 2nd and 3rd columns respectively. Similarly R_1, R_2, R_3 stand for 1st, 2nd and 3rd rows respectively.

Sometimes it is required while evaluating the determinants to alter rows and columns.

Suppose we are required to add twice the constituents of second row and subtract thrice the constituents of third row from the corresponding constituents of first row; we

shall express it as $R_1+2R_2-3R_3$.

Similarly $C_2-aC_3+bC_1$ would mean that subtract a times column no. 3 and add b times column no. 1 from column no. 2.

***Example 6.** Evaluate

$$\Delta=\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}.$$

(Agra 50, 53)

If we expand here directly, the expansion will involve lengthy products and we are likely to commit mistakes. Hence efforts should always be made to reduce the numbers to lesser numbers and to bring in as many zeros as can be possible in any row or column and then expand with that row and column.

Apply C_1-C_2 and C_2-C_3 which in accordance with our notation means that from the constituents of first column subtract the corresponding constituents of 2nd column.

Similarly C_2-C_3 stands for subtracting from the constituents of 2nd column the corresponding constituents of 3rd

$$\therefore \quad \Delta=\begin{vmatrix} 265-240 & 240-219 & 219 \\ 240-225 & 225-198 & 198 \\ 219-198 & 198-181 & 181 \end{vmatrix}$$

$$=\begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}$$

We can further reduce the number in the above determinant by applying

$$C_1-C_2 \text{ and } C_3-10C_2$$

$$\Delta=\begin{vmatrix} 4 & 21 & 219-10(21) \\ -12 & 27 & 198-10(27) \\ 4 & 17 & 181-10(17) \end{vmatrix}$$

$$\Delta=\begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

Apply R_2+3R_1 and R_3-R_1 thus making two zeros in column one

$$\Delta=\begin{vmatrix} 4 & 21 & 9 \\ 0 & 90 & -45 \\ 0 & -4 & 2 \end{vmatrix}=4\begin{vmatrix} 90 & -45 \\ -4 & 2 \end{vmatrix}$$

$$=4\,(180-180)=0$$

EXERCISE No. 1

Note. Before doing the questions, read Art. 20 and Ex. 6 very thoroughly.

(1) Evaluate the following determinants :—

$$\Delta=\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix} \quad \text{Now } \Delta=\begin{vmatrix} 7.6+1 & 1 & 6 \\ 7.4+7 & 7 & 4 \\ 7.2+3 & 3 & 2 \end{vmatrix}$$

Break into sum of two determinants each of which will be zero because of identical lines. $\Delta=0$

(2) $\Delta=\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{vmatrix}$ **where ω is a cube root of unity.**

(Dehli 58 ; Allahabad 27)

We know that if ω is a cube root of unity, then

$1+\omega+\omega^2=0$ and $\omega^3=1$ and $\omega^4=\omega^3.\omega=\omega$.

"Apply $c_1+c_2+c_3$" and thus every constituent of c_1 becomes $1+\omega+\omega^2=0$, $\Delta=0$.

(3) $\Delta=\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$. **(Nagpur 33)**

Proceeding as in Q. 2, $\Delta=0$.

*(4) $\Delta=\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$. Apply c_2+c_3 and take out $a+b+c$ and thus c_1 and c_2 of the new determinant become identical. $\therefore\ \Delta=0$. **(Pb. 35)**

(5) $\Delta=\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix}$. Apply R_3-R_2 and R_2-R_1 and thus R_2 and R_3 of new determinant become identical.

$\therefore\ \Delta=0$

or apply $c_1+c_3-2c_2$; $\therefore\ \Delta=0$.

*(6) Evaluate $\Delta=\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$.

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}-\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$ (Art. 15) **(Calcutta 61)**

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}-\frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$ (Art. 13 Note I

We have multiplied R_1, R_2 and R_3 by a, b, c respectively and hence divided 2nd determinant by abc.

Now take abc common from C_3 of 2nd determinant and make C_3 pass over two columns and thus becoming column No 1. Hence 2nd determinant becomes same as 1st.

$\therefore \quad \Delta=0.$

(7) Evaluate $\Delta=\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$. Apply $C_1+C_2+C_3$ and put $1+\omega+\omega^2=0$ and $\omega^3=1$ then expand. $\Delta=3$.

(8) Evaluate $\Delta=\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$.

(Agra 50 ; Pb. 40 ; Alld. 47)

Apply C_1-3C_3, C_2-2C_3, C_4-4C_3 and expand, thus getting a determinant of 3rd order in which again apply R_2-R_1 and R_3-R_1 and expand with 2nd row. $\therefore \quad \Delta=6$

* (9) Evaluate (a) $\Delta=\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$.

(Sagar 65 ; Alld 56; Banaras 49)

Since [illegible] $1=7$, $1 \quad 2=6$ $1 \quad 2=1$, we apply the following operations

$$R_1-(R_3+R_4), R_2-3R_4, R_3-2R_4$$

[illegible]

$$\therefore\ \Delta=\begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 5 & 7 & 1 & 2 \end{vmatrix}=-1\begin{vmatrix} -8 & -12 & -2 \\ 6 & -2 & 1 \\ -4 & -6 & -1 \end{vmatrix}=0.$$

Because the first and third rows in the above are identical, i e. each constituent of first row is twice of the corresponding constituent of 3rd row.

(b) Evaluate $\begin{vmatrix} 3 & 7 & 9 & 6 \\ 8 & 4 & 5 & 8 \\ 6 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$ *(Agra 56)*

Applying R_2-R_4, we get

$$\Delta=\begin{vmatrix} 3 & 7 & 9 & 6 \\ 2 & 2 & -4 & 0 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$$

Again applying C_2-C_1, C_3+2C_1, we get

$$\Delta=\begin{vmatrix} 3 & 4 & 15 & 6 \\ 2 & 0 & 0 & 0 \\ 7 & 3 & 17 & 5 \\ 6 & -4 & 21 & 8 \end{vmatrix}$$

Expanding with the help of 2nd row,

$$\Delta=-2\begin{vmatrix} 4 & 15 & 6 \\ 3 & 17 & 5 \\ -4 & 21 & 8 \end{vmatrix}$$

Adding R_1 to R_3, we get

$$\Delta = -2 \begin{vmatrix} 4 & 15 & 6 \\ 3 & 17 & 5 \\ 0 & 36 & 14 \end{vmatrix}$$

$$= -2\,[4\,(238-180) - 3\,(210-216)]$$

$$= -2\,(232+18) = -500.$$

Q. 10. Find the value of

(a) $\begin{vmatrix} 30 & 11 & 0 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix}$ (b) $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 44 & 40 \end{vmatrix}$

(Agra M.Sc. 55) (Agra M.Sc. 58)

(a) Apply C_1-2C_2, C_4-3C_2 etc ; $\Delta=9$.

(b) Apply C_1-C_2 and C_2-C_3 and then in the new determinant apply R_2+2R_1 and R_3-3R_1 etc. ; $\Delta=132$.

*(11) Evaluate $\Delta = \begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$

(Lucknow pass 54 ; Gorakhpur 59 ; Agra 32 ; Cal. Hons. 57 ; Patna 49 ; Rajasthan 53)

$$\Delta = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = D + D_1$$

Multiply R_1, R_2, R_3 and R_4 of D_1 by a, b, c and d respectively and hence divide the determinant by $abcd$.

$$D_1=\frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}=\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$

$$=(-1)^3\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}=-D, \quad \therefore \quad D+D_1=0.$$

Hence $\Delta=0$.

(12) Evaluate

$$\begin{vmatrix} 2\alpha & \alpha+\beta & \alpha+\gamma & \alpha+\delta \\ \beta+\alpha & 2\beta & \beta+\gamma & \beta+\delta \\ \gamma+\alpha & \gamma+\beta & 2\gamma & \gamma+\delta \\ \delta+\alpha & \delta+\beta & \delta+\gamma & 2\delta \end{vmatrix}$$ **(Rajputana 60)**

Applying C_1-C_2 and C_2-C_3 and taking $\alpha-\beta$ and $\beta-\gamma$ respectively common from C_1 and C_2 of the new determinant we find that it has two columns identical and hence its value is zero.

(13) Find the value of $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

(Dacca 46 ; Mysore 49 ; Agra 54)

Applying C_4-C_3, C_3-C_2, C_2-C_1, we get

$$\Delta=\begin{vmatrix} 1 & 3 & 5 & 7 \\ 4 & 5 & 7 & 9 \\ 9 & 7 & 9 & 11 \\ 16 & 9 & 11 & 13 \end{vmatrix}$$

Again if we apply C_4-C_3 and C_3-C_2, then columns No. 3 and 4 of the new determinant are identical.

$$\therefore \quad \Delta=0$$

*(14) $$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}=\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}=\begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

(Agra 29 ; Luck. 45)

To prove this you have to change rows into columns and some more movements of lines (Att. 6, 7).

(15) $$\begin{vmatrix} a & b & c & 0 \\ b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{vmatrix}=\begin{vmatrix} -a & b & c & 0 \\ b & -a & 0 & c \\ c & 0 & -a & b \\ 0 & c & b & -a \end{vmatrix}$$

Multiply the first and fourth columns by (-1) and divide the determinant by $(-1)^2$ *i.e.* by 1.

$$\therefore \quad \Delta=\begin{vmatrix} -a & b & c & 0 \\ -b & a & 0 & -c \\ -c & 0 & a & -b \\ 0 & c & b & -a \end{vmatrix}$$

Now take (-1) common from R_2 and R_3 each.

*(17) $\Delta=\begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix}=\begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}$

(Punjab 24)

Proceed as for D_1 of Q. 11 P. 33.

*(18) If $u=ax^2+2bxy+cy^2$ and $u'=a'x^2+2b'xy+c'y^2$, prove that

$$\Delta=\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}=\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

$$=-\frac{1}{y}\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix}$$

Multiply c_1 and c_3 by x any y resp. and hence divide Δ by xy.

$$\therefore\ \Delta=\frac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix}$$ Apply c_1+yc_2 and c_3+xc_2

$$\Delta=\frac{1}{xy}\begin{vmatrix} 0 & -xy & 0 \\ ax+by & b & bx+cy \\ a'x+b'y & b' & b'x+c'y \end{vmatrix}$$ Expand with 2nd column.

$$=\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c' \end{vmatrix}$$

$$=\frac{1}{y}\begin{vmatrix} ax+by & bxy+cy^2 \\ a'x+b'y & b'xy+c'y^2 \end{vmatrix} \quad \text{Now apply } C_2+xC_1$$

$$=\frac{1}{y}\begin{vmatrix} ax+by & ax^2+2bxy+cy^2 \\ a'x+b'y & a'x^2+2b'xy+c'y^2 \end{vmatrix}$$

$$=\frac{1}{y}\begin{vmatrix} ax+by & u \\ a'x+b'y & u' \end{vmatrix}=\frac{1}{y}\begin{vmatrix} ax+by & a'x+b'y \\ u & u' \end{vmatrix}$$

$$=-\frac{1}{y}\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix}$$ We have changed two adjacent rows and hence $-$ive sign.

*(19) $$\Delta=\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}=-(ac-b^2)\ (ax^2+2bxy+cy^2).$$ **(Sagar 49 ; Agra M. Sc. 52 ; Nagpur 61)**

Apply $R_3-(xR_1+yR_2)$ and expand with 3rd row of new determinant thus obtained.

(20) If $u=ax^4+4bx^3+6cx^2+4dx+e,$
$u_{11}=ax^2+2bx+c,$
$u_{12}=bx^2+2cx+d,$
$u_{22}=cx^2+2dx+e,$

prove $$\Delta=\begin{vmatrix} a & b & c & u_{11} \\ b & c & d & u_{12} \\ c & d & e & u_{22} \\ u_{11} & u_{12} & u_{22} & 0 \end{vmatrix}=-u\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

Substitute the values of u_{11}, u_{12} and u_{22} in Δ

Apply $R_4-(x^2R_1+2xR_2+R_3)$ thus getting three zeros in 4th row of the new determinant thus obtained.

(21) Prove that $$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$ **is a perfect square.**

(Agra 41 ; Gauhati Hons. 65)

Take out a, b and c common factors from c_1, c_2 and c_3 respectively and then in the new determinant thus obtained make two zeros. $\Delta=a^2b^2c^2$

(22) **Prove** $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{vmatrix}=4a^2b^2c^2$ (Mysore 48)

Taking a, b, c common from C_1, C_2 and C_3 respectively

$$\Delta=abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$ Apply $C_1+C_2-C_3$.

$$\Delta=abc\begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}=2ab^2c\begin{vmatrix} 0 & c & a+c \\ 1 & b & a \\ 1 & b+c & c \end{vmatrix}$$

Now apply R_2-R_3 and expand etc.

(23) **Prove** $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}=a^3+b^3+c^3-3abc.$ (Agra M.Sc. 57)

Applying $C_1+C_2+C_3$, we get

$$\Delta=(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

$\therefore$ 0 can be written as $0\,(a+b+c)$.

Apply R_3-2R_1 and expand with column No. 1.

(24) **Prove** $\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{c^2+a^2}{b} \end{vmatrix} = 4abc.$ **(Vikram 63)**

Multiply R_1, R_2 and R_3 by c, a and b respectively and hence divide Δ by abc.

$$\therefore \quad \Delta = \frac{1}{abc} \begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

Now make one zero in column no. 2 by $C_2 - C_3$ and then expand.

*(25) **Prove** $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(Mysore 65 ; Delhi 59 ; Rangoon 50 ; Alld. 37)

Apply $C_1 - C_2$, $C_2 - C_3$.

$$\Delta = (a+b+c)^2 \begin{vmatrix} -1 & 0 & 2a \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Now apply $R_1 + R_2 + R_3$ and expand.

(26) **Prove** $\begin{vmatrix} a+b+nc & (n-1)\,a & (n-1)\,b \\ (n-1)\,c & b+c+na & (n-1)\,b \\ (n-1)\,c & (n-1)\,a & c+a+nb \end{vmatrix}$

$= n\,(a$;

Apply $C_1+C_2+C_3$ and take out $n\ (a+b+c)$ common from C_1 of the new determinant and then make two zeros in C_1 and expand.

***27. (a) Prove** $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix} = (ax-by+cz$ **(Cal. Hons. 52 Raj. 57)**

$$\Delta=\frac{1}{a}\begin{vmatrix} 0 & ax & y & z \\ -x & 0 & c & b \\ -y & -ac & 0 & a \\ -z & -ab & -a & 0 \end{vmatrix} \quad \text{Apply } C_2-bC_3+cC_4.$$

$$\Delta=\frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0 & c & b \\ -y & 0 & 0 & a \\ -z & 0 & -a & 0 \end{vmatrix}.$$

Expanding with C_2 and hence $-$ive sign.

$$\therefore \quad \Delta=-\frac{ax-by+cz}{a}\begin{vmatrix} -x & c & b \\ -y & 0 & a \\ -z & -a & 0 \end{vmatrix} \quad \text{Take } (-1) \text{ common from } C_1$$

Now you may expand the determinnant

$$\Delta=\frac{ax-by+cz}{a}\begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}=\frac{ax-by+cz}{a^2}\begin{vmatrix} ax & ac & ab \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

Apply $R_1-bR_2+cR_3$ etc.

(b) $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}=(x-2y+z)^2.$ **(Luck. Pass 55 ; Bombay 58 ; Sagar 50)**

Proceed exactly as in part (*a*) *i.e.* by applying $C_1-2C_2+C_3$ etc.

(28) Prove $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}=abc.$

Make three zeros in C_1 and expand *i.e.* R_2-R_1, R_3-R_1, R_4-R_1.

(29) $\Delta=\begin{vmatrix} 1 & a & a^2 & 0 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix}$ **(Dacca 43)**

Apply $C_1+C_2+C_3+C_4$ and take out the common factor $1+a+a^2$.

$$\Delta=1+a+a^2\begin{vmatrix} 1 & a & a^2 & 0 \\ 1 & 1 & a & a^2 \\ 1 & 0 & 1 & a \\ 1 & a^2 & 0 & 1 \end{vmatrix}.$$

Apply R_2-R_1, R_3-R_1 and R_4-R_1.

$$\Delta=1+a+a^2\begin{vmatrix} 1-a & a-a^2 & a^2 \\ -a & 1-a^2 & a \\ a^2-a & -a^2 & 1 \end{vmatrix}.$$

Apply $R_1-R_2+R_3$

$$=1+a+a^2\begin{vmatrix} 1+a^2-a & -(1+a^2-a) & 1+a^2-a \\ -a & 1-a^2 & a \\ a^2-a & -a^2 & 1 \end{vmatrix}$$

Take out $1+a^2-a$ common from R_1

$$\Delta=[(1+a^2)^2-a^2]\begin{vmatrix} & -1 & 1 \\ -a & 1-a^2 & a \\ a^2-a & -a^2 & 1 \end{vmatrix}.$$

Now apply C_2+C_1 and C_3-C_1 and then expand.

*(30) **Prove**

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}=(1+a^2+b^2)^3. \qquad \text{(Pb. 34)}$$

Apply C_1-bC_3 and C_2+aC_3 and take out $(1+a^2+b^2)$ each common from C_1 and C_2 of the new determinant thus obtained.

$$\therefore\quad \Delta=(1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}. \text{ Now expand.}$$

***(31) Prove**

$$\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix} = abc\,(a^2+b^2+c^2)^3$$

Taking a, b, c common from C_1, C_2 and C_3 respectively,

$$= abc \begin{vmatrix} a^2-b^2-c^2 & 2b^2 & 2c^2 \\ 2a^2 & b^2-c^2-a^2 & 2c^2 \\ 2a^2 & 2b^2 & c^2-a^2-b^2 \end{vmatrix}$$

Apply R_2-R_1 and R_3-R_1 and take $(a^2+b^2+c^2)$ each common from R_2 and R_3 of the new determinant thus obtained and then expand.

(32) Prove $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$

(Cal. Hons. 55 ; Punjab 64 ; Banaras 57)

$$\Delta = \frac{1}{a} \begin{vmatrix} a^2 & b & c & d \\ -ab & a & -d & c \\ -ac & d & a & -b \\ -ad & -c & b & a \end{vmatrix}$$

Apply $C_1+bC_2+cC_3+dC_4$.

$$= \frac{1}{a} \begin{vmatrix} a^2+b^2+c^2+d^2 & b & c & d \\ 0 & a & -d & c \\ 0 & d & a & -b \\ 0 & -c & b & a \end{vmatrix}$$

$$=\frac{a^2+b^2+c^2+d^2}{a}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}.\ \text{Now expand.}$$

*(33) **Prove**

$$\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}=a^4.$$

Apply R_4-R_3, R_3-R_2, R_2-R_1.

$$\Delta=a^3\begin{vmatrix} 1 & a+b & a+b+c \\ 1 & 2a+b & 3a+2b+c \\ 1 & 3a+b & 6a+3b+c \end{vmatrix}.$$

Again apply R_3-R_2, R_2-R_1 and then expand.

*(34) Find the value of

$$\Delta=\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}.$$

(Allahabad 51 : Agra 43)

Apply R_5-R_4, R_4-R_3, R_3-R_2, R_2-R_1 and expand with 1st column.

$$\Delta=\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 34 \end{vmatrix}.$$

Again apply R_4-R_3, R_3-R_2, R_2-R_1 and expand.

$$\Delta=\begin{vmatrix} 1 & 3 & 6 \\ 1 & 4 & 10 \\ 1 & 5 & 14 \end{vmatrix}$$

Again apply R_3-R_2, R_2-R_1.

$$\Delta=\begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}=0.$$

35. (a) Prove that

$$\Delta=\begin{vmatrix} \sin^2 A & \sin A\cos A & \cos^2 A \\ \sin^2 B & \sin B\cos B & \cos^2 B \\ \sin^2 C & \sin C\cos C & \cos^2 C \end{vmatrix}=-\sin(A-B)\sin(B-C)\sin(C-A)$$

where $A+B+C=\pi$.

(Punjab 60)

Applying C_3+C_1, we get

$$\Delta=\begin{vmatrix} \sin^2 A & \frac{1}{2}\sin 2A & 1 \\ \sin^2 B & \frac{1}{2}\sin 2B & 1 \\ \sin^2 C & \frac{1}{2}\sin 2C & 1 \end{vmatrix}, \quad \because \sin^2 x+\cos^2 x=1.$$

Apply R_2-R_1 and R_3-R_1 and note that

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

and $$\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B).$$

$$\therefore \Delta=\begin{vmatrix} \sin^2 A & \frac{1}{2}\sin 2A & 1 \\ \sin(B+A)\sin(B-A) & \cos(B+A)\sin(B-A) & 0 \\ \sin(C+A)\sin(C-A) & \cos(C+A)\sin(C-A) & 0 \end{vmatrix}.$$

Now $\sin(A+B)=\sin(\pi-C)=\sin C$

and $\cos(A+B)=\cos(\pi-C)=-\cos C$.

$$\Delta=\sin(B-A)\sin(C-A)\begin{vmatrix} \sin C & -\cos C \\ \sin B & -\cos B \end{vmatrix}$$

$$=-\sin(A-B)\sin(B-C)\sin(C-A).$$

(Vikram B.Sc. 59)

35. (b) Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix}=-4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}.$$

Apply C_1-C_2, C_2-C_3 and expand

***(36) Prove the following identities :—**

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}=(x-y)(y-z)(z-x)\times(xy+yz+zx)$$

(Delhi 40. 58 ; Cal. 46 ; Pb. 48, 59 ; Agra 62 ; Luck. 49 Supp. ; Utkal 50, 63)

Multiply C_1, C_2 and C_3 and x, y and z respectively and hence divide the determinant by xyz.

$$\therefore\ \Delta=\frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}=\frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$=(-1)^2\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}\text{(Art. 8)}=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Now proceeding as in Ex. 1, 2, P. 19, 20, *i.e.* by making two zeros in first row,

$$\Delta=\begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & y+z \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}.$$

Again applying C_2-C_1, we get

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & (z^2-x^2)+y(z-x) \end{vmatrix}.$$

$$=(x-y)(y-z)(z-x)\begin{vmatrix} x+y & 1 \\ x^2+xy+y^2 & x+y+z \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)\left[(x+y)^2+z(x+y)-(x^2+xy+y^2)\right]$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx).$$

(37) $\Delta=\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}=-(a+b+c)\quad(a-b)\quad(b-c)\times(c-a).$ **(Patna 23)**

Apply C_2+C_1 and then interchange C_1 and C_2

$$=-(a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ etc. from Ex. 1 P. 19.

(38) Simplify

$$\begin{vmatrix} a^3 & a & 1 \\ b^3 & & 1 \\ c^3 & c & 1 \end{vmatrix}\div\begin{vmatrix} a^2 & c & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$ or $D\div D_1$

Now by Ex. 1 and 2 pages 19 and 20 factors of D and D_1 are $(a-b)(b-c)(c-a)(a+b+c)$ and $(a-b)(b-c)(c-a)$ respectively. $\therefore\ D-D_1=a+b+c$.

(39) Factorize

$$\Delta=\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$=abc\,(a-b)\,(b-c)\,(c-a)$ Ex. 1 Art. 17 page 19.

Note. See just now Q 6, 7 of Misc. Exercise.

(40) If x, y and z are all different and given that

$$\Delta=\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0, \text{ prove } 1+xyz=0$$

(Nagpur 30 ; Annamalai 38 : Agra 64)

$$\Delta=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Hence by Ex. 1 Art. 17 page 19.

$$\Delta=(x-y)(y-z)(z-x)(1+xyz)=0.$$

But because x, y and z are all different.

$\therefore$ neither of $x-y$, $y-z$ and $z-x$ is zero.

Hence $\qquad 1+xyz=0$.

(41) If a, b and c are all different and given that

$$\Delta=\begin{vmatrix} a & a^2 & a^4-1 \\ b & b^2 & b^4-1 \\ c & c^2 & c^4-1 \end{vmatrix}=0, \text{ then } abc\,(bc+ca+ab)=a+b+c.$$

Proceed as above and use Q. 36 P. 46 and Ex. 2 Art. 17 P. 19.

(42) Prove that

$$\Delta=\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}=(a+b+c+d)(a+d-b-c)(a+c-b-d)(a+b-c-d)$$

Ortho-symmetric determinant.

(Luck. 46 ; Pb. 42, 46 ; Nag. 50 ; Cal. Hons. 60 ; Delhi Hons. 65)

Applying $C_1+C_2+C_3+C_4$, the new column number one shall give $a+b+c+d$ as a factor of Δ.

Similarly applying $C_1+C_2-C_3-C_4$, we find that $a+b-c-d$ will also be a factor.

Again applying $C_1+C_3-C_2-C_4$, we get $a+c-b-d$ as a factor

and applying $C_1+C_4-C_2-C_3$, we get $a+d-b-c$ as a factor

$\therefore$ $(a+b+c+d)(a+b-c-d)(a+c-b-d)(a+d-b-c)$ is a factor of fourth degree of Δ which is also of the same degree. Hence any other factor may be some constant K whose value is found to be unity by comparing the coefficient of a^4 the diagonal term.

*(43) **(a) Prove that**

$$\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}=4\ (b+c)(c+a)(a+b).$$

(Alld. 37 ; Patna 42 ; Luck. 46 Supp. ; Utkal 48)

Expand directly or refer Ex. 1, 2. P. 19, 20, 2nd Method.

If we put $a+b=0$ *i.e.* $b=-a$ in the given determinant and expand it, it vanishes, showing that $a+b$ is a factor. Similarly $b+c$ and $c+a$ are also its factors.

$$\therefore \quad \Delta=k\,(a+b)(b+c)(c+a).$$

In order to find k, put $a=0$, $b=1$, $c=1$ in both sides giving $8=2k$; $\therefore\ k=4$.

(b) Prove that

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3$$

Applying C_1+C_3 and taking $a+b+c$ common,

$$\Delta=(a+b+c)\begin{vmatrix} 1 & a-b & a \\ 1 & b-c & b \\ 1 & c-a & c \end{vmatrix}$$

Now make two zeros by R_2-R_1 and R_3-R_1 and expand.

$$\therefore \quad \Delta=(a+b+c)\{-(a^2+b^2+c^2-ab-bc-ca)\}$$
$$=-(a^3+b^3+c^3-3abc) \text{ etc.}$$

*(44) $$\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = -(x+y+z)(x+y-z)(z+x-y) \times (y+z-x)$$

(Agra 33 ; Rajputana 52)

Proceeding exactly as in Q. 42 P. 49, we get

$$\Delta=\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = K(x+y+z)(y+z-x)(z+x-y)(x+y-z) \ldots (1)$$

It is easy to find K, by comparing the coefficients of x^4 in both sides of (A).

Expanding Δ only for x,

$$-x\begin{vmatrix} x & z & y \\ y & 0 & x \\ z & x & 0 \end{vmatrix} + \text{two other determinants which shall not give } x^4$$

$= -x\,[x\,(0-x^2)+-+-] = x^4 +$ other terms which shall not be of x^4.

$\therefore$ the coefficient of x^4 is unity whereas it is $-K$ on the R. S. of (A).

$\therefore \quad K=-1.$ Hence proved.

*(45) $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = -(x+y+z)\,(x+y-z)\,(z+x-y)\,(y+z-x)$$

(Rajputana 49)

Applying C_3-C_2 and C_4-C_2, we get

$$\Delta = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & -z^2 & x^2-z^2 \\ 1 & y^2 & x^2-y^2 & -y^2 \end{vmatrix} = -\begin{vmatrix} 1 & z^2 & y^2 \\ 1 & -z^2 & x^2-z^2 \\ 1 & x^2-y^2 & -y^2 \end{vmatrix}.$$

Again apply R_2-R_1 and R_3-R_1.

$$\Delta = -\begin{vmatrix} 2z^2 & y^2+z^2-x^2 \\ y^2+z^2-x^2 & 2y^2 \end{vmatrix} = (y^2+z^2-x^2)^2-4y^2z^2$$

$=(y^2+z^2-x^2+2yz)\,(y^2+z^2-x^2-2yz)$

$=\{(y+z)^2-x^2\}\,\{(y-z)^2-x^2\}$ etc. as given.

*(46) Prove that

$$\begin{vmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where capital letters denote the minors of the corresponding small letters in the determinant on the right. (Agra B. Sc. 66)

Denote the two determinants on the left and right by D and D' respectively ; then

$$DD' = \begin{vmatrix} a_1A_1-b_1B_1+c_1C_1 & a_2A_1-b_2B_1+c_2C_1 & a_3A_1-b_3B_1+c_3C_1 \\ -a_1A_2+b_1B_2-c_1C_2 & -a_2A_2+b_2B_2-c_2C_2 & -a_3A_2+b_3B_2-c_3C_2 \\ a_1A_3-b_1B_3+c_1C_3 & a_2A_3-b_2B_3+c_2C_3 & a_3A_3-b_3B_3+c_3C_3 \end{vmatrix}$$

which by the result of article 11 on P. 10–11.

$$= \begin{vmatrix} D' & 0 & 0 \\ 0 & D' & 0 \\ 0 & 0 & D' \end{vmatrix} = D'^3.$$

$$\therefore \quad DD' = D'^3$$

or $\qquad D = D'^2.$

Similarly with the help of article no. 11 P. 12–13 we can prove

*(47) $$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2.$$

where capital letters denote the cofactors of the corresponding small letters of the determinant on the right.

(Bihar 66, Pb. 31. Mādras 38, Annamalai 38, Luck. 42, Suppl, Calcutta 55, 61, Delhi Hons 63, Agra 42, Dacca 65, Sagar 65, Aligarh 49, Bom 51, Nag. 51, 55, Alld. 52).

Note, See also Q. 67 P. 64 and Q 28 P. 96.

$$=\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$ which on multiplying is

$$=\begin{vmatrix} \Sigma x^2 & \Sigma xy & \Sigma xy \\ \Sigma xy & \Sigma x^2 & \Sigma xy \\ \Sigma xy & \Sigma xy & \Sigma x^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix},$$

$$\because \quad \Sigma x^2 = x^2+y^2+z^2 = r^2$$

and $$\Sigma xy = xy+yz+zx = u^2.$$

(50) If ω is a cube root of unity, then show that square of

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix}.$$

(Alld. 27 ; Travancore 43 ; Nagpur 50)

and hence find the value of the determinant on the L. H. S

Replace ω^3 by 1 and remember $1+\omega+\omega^2=0$. Let us denote the determinant in L. H. S. by Δ.

$$\therefore \quad \Delta^2 = \begin{vmatrix} 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix} \times \begin{vmatrix} 1 & \omega & \omega^3 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix}.$$

First constituent of first row in product determinant

$$=1+\omega^3+\omega^4+1=1+\omega^2+\omega+1=1+0=1.$$

In a similar manner find other constituent of first row and other rows and we get

$$\Delta^2=\begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix}$$

Applying R_2-R_1, R_3+2R_1 and R_4-R_1 and then expanding, we find that $\Delta^2=-27$, $\therefore\ \Delta=3\sqrt{-3}$.

(51) Find the value of

$$\Delta=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

(Pb. 60)

where $l_1^2+m_1^2+n_1^2=1$, etc. and $l_1l_2+m_1m_2+n_1n_2=0$, etc.

$$\Delta^2=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\times\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$=\begin{vmatrix} \Sigma l_1^2 & \Sigma l_1l_2 & \Sigma l_1l_2 \\ \Sigma l_1l_2 & \Sigma l_2^2 & \Sigma l_2l_3 \\ \Sigma l_1l_3 & \Sigma l_2l_3 & \Sigma l_3^2 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=1.$$

$$\therefore\ \Delta=\pm1.$$

***(52) Prove that**

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}\times\begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & \alpha+i\beta \end{vmatrix}$$

can be written in the form $\begin{vmatrix} A-iB & C-iD \\ -C-iD & A+iB \end{vmatrix}$.

(Agra 28, 44 ; Travancore 47 ; Cal. Hons. 60)

On multiplying, the first constituent of first row in

determinant will be

$$\begin{aligned}&(a+ib)(\alpha-i\beta)+(c+id)(\gamma-i\delta)\\&=\underbrace{a\alpha+b\beta+c\gamma+d\delta)}_{A}-i\underbrace{(a\beta-b\alpha+c\delta-d\gamma)}_{B}\\&=\quad A\quad -i\quad B\end{aligned}$$

Similarly 2nd constituent is

$$\begin{aligned}&(a+ib)(-\gamma-i\delta)+(c+id)(\alpha+i\beta)\\&=\underbrace{-a\gamma+b\delta+c\alpha-d\beta}_{C}-i\underbrace{(a\delta+c\gamma-c\beta-d\alpha)}_{D}\\&=\quad C\quad -i\quad D\end{aligned}$$

Similarly the constituents of 2nd row can be easily found out to be

$$-C-iD \text{ and } A+iB$$

$$\therefore \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & \alpha+i\beta \end{vmatrix} = \begin{vmatrix} A-iB & C-iD \\ -C-iD & A+iB \end{vmatrix}$$

where A, B, C and D have the values written above.

Expanding these determinants, we get

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=A^2+B^2+C^2+D^2$$

which shows that the product of two sums each of four squares can be expressed as the sum of four squares.

Hence express

$$(9^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$$

as the sum of four squares.

Let $a=9$, $b=2$, $c=3$, $d=4$ and $\alpha=5$, $\beta=6$, $\gamma=7$, $\delta=8$.

Then $\qquad A=a\alpha+b\beta+c\gamma+d\delta=110.$

Similarly $\qquad B=40$, $C=56$ and $D=48$.

$$\therefore \quad (9^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$$

$$=(110^2+40^2+56^2+48^2)$$

(53) Express as a determinant

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\times\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$=\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

(54) Prove that

$$\begin{vmatrix} x^2+y^2+a^2 & 2ax+xy & 2ay+x^2 \\ 2ax+xy & a^2+2x^2 & 2ax+xy \\ 2ay+x^2 & 2ax+xy & x^2+y^2+a^2 \end{vmatrix}=\begin{vmatrix} a & x & y \\ x & a & x \\ y & x & a \end{vmatrix}$$

After multiplying $\begin{vmatrix} a & x & y \\ x & a & x \\ y & x & a \end{vmatrix}$ by itself, we get determinant on L.H.S.

Hence proved.

(55) Prove that $\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$

is a multiple of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ and find the other factor. (Pb. 46)

It is easy to see that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix}, \text{ etc,}$$

Hence the other factor is $\begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix}$ *i.e.* 13.

Solve the equations

(56) $\Delta = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0.$ Put $x=a$ and b and you find that Δ vanishes because of identical lines. Hence a and b are roots of Δ which is of second degree in x.

(57) $$\Delta = \begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0.$$

Apply $R_1 - 2R_3$ and $R_2 - 3R_3$.

$$\Delta = \begin{vmatrix} 1 & -17 & -16 \\ -10 & -25 & -23 \\ 7-x & 14 & 13 \end{vmatrix} = 0.$$

Again apply C_3-C_2 and then make two zeros in C_3 of the determinant and expanding, we get $x=4$.

(58) $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}=0$ **(Agra 35, 51; Luck. 48)**

Applying R_2-R_1 and R_3-R_1, we get

$$\Delta=\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix}=0.$$

$$\therefore \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}=0.$$

Applying C_3-2C_2 and C_2-3C_1 and thus making two zeros in row No. 2 of the new determinant and expanding, we get $x=4$.

*(59) $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+2 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix}=0.$ **(Agra 33; Luck. 50)**

Apply $C_2-\frac{3}{2}C_1$ and C_3-2C_1.

$$\Delta=\begin{vmatrix} 4x & 2 & 1 \\ 6x+2 & -1 & -4 \\ 8x+1 & -\frac{3}{2} & 0 \end{vmatrix}=0.$$

Now expanding with third column, we find $x=-\frac{10}{89}$.

*(60) $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix}=0.$ **(Pb. 37, 49)**

Applying R_3-R_2, R_2-R_1, we get

$$\Delta=(x+1)(x+2)\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix}$$

Now apply C_3-C_2 and C_2-C_1 etc. $\therefore\ x=-2, -1, -1.$

***(61) (a) Show that $x=2$ is a root of**

$$\Delta=\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}=0$$

and solve it completely. **(Agra 48; Raj. 48)**

Applying R_1-R_2 and taking $(x-2)$ common, we get

$$\Delta=(x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

Now apply C_2-3C_1, C_3+C_1 and expand.

$$\Delta=(x-2)\begin{vmatrix} -3x-6 & x-1 \\ 2x+9 & x-1 \end{vmatrix}=(x-2)(x-1)(-5x-15)=0.$$

$\therefore\ x=1, 2, -3.$

(b) Given $a+b+c=0$, solve

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0.$$

(Burdwan 64; Karnatak 62; Agra 61; Bombay 48; Venkateswara 59)

Applying $C_1+C_2+C_3$ and putting $a+b+c=0$ and taking $-x$ common, we get

$$-x\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}=0.$$

Applying R_2-R_1 and R_3-R_1,

(d) Solve the following equation

$$\begin{vmatrix} 1 & 2 & -3 & (x-5) \\ 1 & 4 & 9 & (x-5)^2 \\ 1 & 8 & -27 & (x-5)^3 \\ 1 & 16 & 81 & (x-5)^4 \end{vmatrix}=0$$

(Lucknow 65)

Apply $R_4-R_3,\ R_3-R_2,\ R_2-R_1$

$$\Delta=\begin{vmatrix} 1 & 2 & -3 & (x-5) \\ 0 & 2 & 12 & (x-5)\ (x-5) \\ 0 & 4 & -36 & (x-5)^2\ (x-6) \\ 0 & 8 & 108 & (x-5)^3\ (x-6) \end{vmatrix}=0$$

Expanding with first column and taking out common factors we get

$$\Delta=2\times 12\times (x-5)\ (x-6)\begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & (x-5) \\ 4 & 9 & (x-5)^2 \end{vmatrix}=0$$

Apply C_2-C_1 and C_3-C_1

$$\Delta=24\ (x-5)\ (x-6)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -5 & x-7 \\ 4 & 5 & (x-7)\ (x-3) \end{vmatrix}=0$$

$$\Delta=24\ (x-5)\ (x-6)\times 5\ (x-7)\begin{vmatrix} -1 & 1 \\ 1 & x-3 \end{vmatrix}=0$$

$$120\ (x-5)\ (x-6)\ (x-7)\ (-x+3-1)=0$$

$$=-120\ (x-2)\ (x-5)\ (x-6)\ (x-7)=0$$

$$=2,\ 5,\ 6,\ 7$$

(c) Show that the equations

$$a_1x^2+b_1x+c_1=0$$
$$a_2x^2+b_2x+c_2=0$$

possess a common root only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$ **(Lucknow 65)**

If α be the common root then

$$a_1\alpha^2+b_1\alpha+c_1=0$$
$$a_2\alpha^2+b_2\alpha+c_2=0$$
$$=\frac{\alpha^2}{b_1c_2-b_2c_1}=\frac{\alpha}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$
$$\therefore \quad (c_1a_2-c_2a_1)^2=(b_1c_2-b_2c_1)\,(a_1b_2-a_2b_1)$$

or
$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

62 Solve :— $x+y+z=1,$
$$ax+by+cz=K,$$
$$a^2x+b^2y+c^2z=K^2.$$

(Delhi Hon,s 63 ; Pb. 61)

By art. 19, $x\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ K & b & c \\ K^2 & b^2 & c^2 \end{vmatrix}.$

$\therefore \quad x\,(a-b)\,(b-c)\,(c-a)=(K-b)\,(b-c)\,(c-K)$. [Ex. 1, § 17].

$$\therefore \; x=\frac{(K-b)\,(b-c)\,(c-K)}{(a-b)\,(b-c)\,(c-a)}=\frac{(K-b)\,(c-K)}{(a-b)\,(c-a)} \qquad \text{(P. 19)}$$
$$=\frac{(K-b)\,(K-c)}{(a-b)\,(a-c)}.$$

Similarly write the values of y and z by symmetry.

(63) Solve :— $ax+by+cz=K,$
$$a^2x+b^2y+c^2z=K^2, \qquad \text{(by Art. no. 19)}$$
$$a^3x+b^3y+c^3z=K^3.$$

$$x\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=\begin{vmatrix} K & b & c \\ K^2 & b^2 & c^2 \\ K^3 & b^3 & c^3 \end{vmatrix}$$

or $xabc\,(a-b)\,(b-c)\,(c-a)=Kbc\,(K-b)\,(b-c)\,(c-K)$. (Ex. 1 Art. 17)

$\therefore\ x=\dfrac{K\,(K-b)\,(K-c)}{a\,(a-b)\,(a-c)}$. Similarly write the values of y and z by symmetry.

(64) Solve :—

$$x+y+z+u=1,$$
$$ax+by+cz+du=K,$$
$$a^2x+b^2y+c^2z+d^2u=K^2, \qquad \text{(by Art. 19)}$$
$$a^3x+b^3y+c^3z+d^3u=K^3.$$

$$u.\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & K \\ a^2 & b^2 & c^2 & K^2 \\ a^3 & b^3 & c^3 & K^3 \end{vmatrix}$$

or $$u\,(a-b)\,(a-c)\,(a-d)\,(b-c)\,(b-d)\,(c-d)$$
$$=(a-b)\,(a-c)\,(a-K)\,(b-c)\,(b-K)\,(c-K).$$

$\therefore\ u=\dfrac{(a-K)\,(b-K)\,(c-K)}{(a-d)\,(b-d)\,(c-d)}$ or $\dfrac{(K-a)\,(K-b)\,(K-c)}{(d-a)\,(d-b)\,(d-c)}$.

Similarly write down the values of x, y and z by symmetry [See Q. 1, P. 69 for factors].

(65) Solve :—

$$x+y+z+u=K,$$
$$ax+by+cz+du=K^2\ ;$$
$$a^2x+b^2y+c^2z+d^2u=K^3,$$
$$a^3x+b^3y+c^3z+d^3u=K^4.$$

Proceeding as above, $u=\dfrac{K(K-a)(K-b)(K-c)}{(d-a)(d-b)(d-c)}$.

(66) Solve :— $x+2y+3z=6$,

$$2x+4y+z=7, \qquad [x=y=z=1]$$

$$3x+2y+9z=14.$$

(57) If $(f^2-bc)\,x+(ch-fg)\,y+(bg-hf)\,z=0$, . (1)

$(ch-fg)\,x+(g^2-ca)\,y+(af-gh)\quad z=0$, . (2)

$(bg-hf)\,x+(af-gh)\,y+(h^2-ab)\quad z=0$. ...(3)

then show that $abc+2fgh-af^2-bg^2-ch^2=0$.

In order that the three given equations may hold, then by artical No. 2 the eliminant

$$\begin{vmatrix} f^2-bc & ch-fg & bg-hf \\ ch-fg & g^2-ca & af-gh \\ bg-hf & af-gh & h^2-ab \end{vmatrix}=0. \quad ..(A)$$

Consider $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Minor of $a=A=bc-f^2$. Similarly $B=ac-g^2$, $C=ab-h^2$ $F=af-gh$, $G=hf-bg$ and $H=ch-fg$.

Hence determinant (A) becomes

$$\begin{vmatrix} -A & H & -G \\ H & -B & F \\ -G & F & -C \end{vmatrix}=0$$

or

$$\begin{vmatrix} A & -H & G \\ -H & B & -F \\ G & -F & C \end{vmatrix}=0 \text{ or by Q. 46, P.52.}$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2=0 \; ; \quad \therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

Expanding, $abc+2fgh-af^2-bg^2-ch^2=0$.

(68) Find the condition that $lx+my+nz=0$ may be satisfied by (a_r, b_r, c_r) where $r=1, 2, 3$ and show that it is the same as the condition that three equations $a_rx+b_ry+c_rz=0$ may be satisfied by (l, m, n).

Since (a_r, b_r, c_r) for $r=1, 2, 3$ satisfy $lx+my+nz=0$.

$$\therefore \quad a_1l+b_1m+c_1n=0,$$
$$a_2l+b_2m+c_2n=0,$$
$$a_3l+b_3m+c_3n=0.$$

Eliminating l, m, n by article No. 2, we get the required condition as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0. \qquad \cdots(A)$$

Again since $a_rx+b_ry+c_rz=0$ where $r=1, 2, 3$ are satisfied by l, m, n,

$$\therefore \quad a_1l+b_1m+c_1n=0,$$
$$a_2l+b_2m+c_2n=0,$$
$$a_3l+b_3m+c_3n=0.$$

Again eliminating l, m, n by article No. 2, we get the condition as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

which is the same as condition (A).

(69) Find λ when

$$ax+hy+g=0,$$
$$hx+by+f=0,$$
$$gx+fy+c=\lambda$$

or

$$ax+hy+g=0,$$
$$hx+by+f=0,$$
$$gx+fy+c-\lambda=0.$$

Eliminating x, y from above set of equations, we get

$$\begin{vmatrix} a & h & g+0 \\ h & b & f+0 \\ g & f & c-\lambda \end{vmatrix}=0$$

or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}+\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & -\lambda \end{vmatrix}=0.$$

$$(abc+2fgh-af^2-bg^2-ch^2)-\lambda(ab-h^2)=0.$$

$$\therefore \quad \lambda=\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}$$

(70) Find K where

$$l_1x+m_1y+n_1z+p_1=k,$$
$$l_2x+m_2y+n_2z+p_2=0,$$
$$l_3x+m_3y+n_3z+p_3=0,$$
$$l_4x+m_4y+u_4z+p_4=0.$$ Proceed as above.

$$K=\begin{vmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{vmatrix}\div\begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{vmatrix}$$

*(71) **Eliminate** a, b **and** c **from**

$$x=\frac{a}{b-c},\ y=\frac{b}{c-a},\ z=\frac{c}{a-b}. \qquad \text{(Pb. 36)}$$

$$\therefore\ a-bx+cx=0,$$
$$-ay-b+cy=0,$$
$$az-bz-c=0.$$

Eliminating a, b, c by article 2, we get

$$\begin{vmatrix} 1 & -x & x \\ -y & -1 & y \\ z & -z & -1 \end{vmatrix}=0. \qquad \text{Expanding, we get } 1+xy+yz+zx=0.$$

*(72) **Having given the equations**

$$x=cy+bz,\ y=az+cx,\ z=bx+ay$$

where x, y, z **are not all zero, prove that**

$$a^2+b^2+c^2+2abc=1,$$

$$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}. \qquad \text{(Pb. 45, 64)}$$

Rewriting the above equations and eliminating x, y, z, we get

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}=0$$

or $$a^2+b^2+c^2+2abc=1 \text{ (on expansion)} \qquad \ldots(1)$$

Again solving the first two equations by the method of cross-multiplication, we get

$$\frac{x}{-ac-b}=\frac{y}{-bc-a}=\frac{z}{-1+c^2}.$$

Squaring, $\dfrac{x^2}{a^2c^2+b^2+2abc}=\dfrac{y^2}{a^2+b^2c^2+2abc}=\dfrac{z^2}{(1-c^2)^2}.$

Using relation (1), we get

$$\frac{x^2}{1-a^2-c^2+a^2c^2}=\frac{y^2}{1-b^2-c^2+b^2c^2}=\frac{z^2}{(1-c^2)^2}$$

or
$$\frac{x^2}{(1-a^2)(1-c^2)}=\frac{y^2}{(1-b^2)(1-c^2)}=\frac{z^2}{(1-c^2)^2}.$$

Cancelling $(1-c^2)$ throughout the denominator, we get the required result.

MISCELLANEOUS EXERCISE

Q. 1. Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

(Agra M. Sc. 57 ; Pb. 39, 44 ; Sagar 66)

Putting $a=b$ and so on we find the following factors :-

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

of sixth degree of the given determinant which too is of the same degree. Hence. if there can be any other factor that must be a constant K ;

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = K(a-b)(a-c)(a-d)(b-c)(b-d)\times(c-d)$$

Comparing the coefficiect of diagonal term bc^2d^3 we find $K=1$.

Q. 2. (a) If $\alpha, \beta, \gamma, \delta$ be the roots of a biquadratic and S_r stands for $\alpha^r+\beta^r+\gamma^r+\delta^r$, then prove that

$$\begin{vmatrix} S_0 & S_1 & S_2 & S_3 \\ S_1 & S_2 & S_3 & S_4 \\ S_2 & S_3 & S_4 & S_5 \\ S_3 & S_4 & S_5 & S_6 \end{vmatrix} = (\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2 \times(\beta-\delta)^2(\gamma-\delta)^2$$

Replacing a, b, c and d by α, β, γ and δ respectively in Q. 1, we get

$$\Delta^2=(\alpha-\beta)^2\,(\alpha-\gamma)^2\,(\alpha-\delta)^2\,(\beta-\gamma)^2\,(\beta-\delta)^2\,(\gamma-\delta)^2$$

where $$\Delta^2=\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}$$

By actual multiplication of determinants and using the given notation $S_r=\alpha^r+\beta^r+\gamma^r+\delta^r$ *i e.* $S_2=\alpha^2+\beta^2+\gamma^2+\delta^2$ and $S_0=\alpha^0+\beta^0+\gamma^0+\delta^0=1+1+1+1=4$ etc., we get the given determinant in L. H. S. Hence proved.

(b) Prove that

$$\begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix} \text{or} \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

$=(a-b)^2\,(b-c)^2\,(c-a)^2$. **(I. A. S. 58 ; Banaras 38)**

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \text{etc}$$

[Refer Ex. 1 P. 19]

Q. 3. (a) Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

$=(a-b)\,(a-c)\,(a-d)\,(b-c)\,(b-d)\,(c-d)\,(a+b+c+d)$.

The first six factors are found as in Q 1 and the last factor $(a+b+c+d)$ is found by symmetery. The remaining factor which will be constant K is easily found to be unity by comparing the coefficient of diagonal term bc^2d^3 etc.

(b) Solve for x

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & \beta & \beta^2 & \beta^3 \\ 1 & \gamma & \gamma^2 & \gamma^3 \end{vmatrix}$$

(Agra M.Sc. 40 : I.A.S 54)

$$\Delta=-(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma)=0$$

$$\therefore \quad x=\alpha,\ \beta,\ \gamma, \text{ when } \alpha\neq\beta\neq\gamma.$$

(c) Show that each of the coefficient of any equation can be expressed in terms of the roots as the quotients of two determinants.

Consider an equations of 3rd degree

$$\Delta=\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix}$$

If we put $x=\alpha$, β, γ or or $\alpha=\beta$ or γ or $\beta=\gamma$, then the two rows become identical and hence the determinant vanishes. Therefore.

$$(x-\alpha)(x-\beta)(x-\gamma)(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$$

are its factors of sixth degree. If there be any other factor it is constant and let it be k.

Comparing the coefficient of diagonal term $x^3\alpha^2\beta$ on both sides we find that $k=1$.

$\therefore \quad \Delta=(x-\alpha)(x-\beta)(x-\gamma)(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$

or $\quad \Delta=-(x-\alpha)(x-\beta)(x-\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

But $\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}=-(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$. [Ex. 1 P. 19]

$$\therefore \quad \Delta=\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}(x-\alpha)(x-\beta)(x-\gamma).$$

$$\text{or} \quad \Delta=\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}(x^3+p_1x^2+p_2x+p_3),$$

where $x^3+p_1x^2+p_2x+p_3=0$ is an equation whose roots are α, β, γ.

Now expanding Δ in terms of first row, we get

$$x^3\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}-x^2\begin{vmatrix} \alpha^3 & \alpha & 1 \\ \beta^3 & \beta & 1 \\ \gamma^3 & \gamma & 1 \end{vmatrix}$$

$$+x\begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix}-\begin{vmatrix} \alpha^3 & \alpha^2 & \alpha \\ \beta^3 & \beta^2 & \beta \\ \gamma^3 & \gamma^2 & \gamma \end{vmatrix}$$

$$=\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}(x^3+p_1x^2+p_2x+p_3).$$

Dividing throughout by $\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}$.

and comparing the coefficients of like powers of x, we get

$$p_1 = -\begin{vmatrix} \alpha^3 & \alpha & 1 \\ \beta^3 & \beta & 1 \\ \gamma^3 & \gamma & 1 \end{vmatrix} \div \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \beta & 1 \end{vmatrix}$$

$$p_2 = \begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} \div \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}$$

$$p_3 = \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha \\ \beta^2 & \beta^3 & \beta \\ \gamma^3 & \gamma^2 & \gamma \end{vmatrix} - \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}$$

Above shows that the coefficients p_1. p_2. p_3 can be exprersed as the quotient of two determinants.

***Q. 4. Prove that**

$$\Delta = \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix}$$

$$=(a-b)\,(a-c)\,(a-d)\,(b-c)\,(b-d)\,(c-d).$$

(Indore B. Sc. 66 ; Agra 22 ; Pb. 51)

Put $bc+ad=A$, $ca+bd=B$ and $ab+cd=C$.

$$\therefore \quad \Delta = \begin{vmatrix} 1 & A & A^2-2abcd \\ 1 & B & B^2-2abcd \\ 1 & C & C^2-2abcd \end{vmatrix}$$

$$\therefore \quad \Delta = a'^2b'^2c'^2 \begin{vmatrix} BC & -A & 1 \\ CA & -B & 1 \\ AB & -C & 1 \end{vmatrix} \quad \begin{array}{l} \text{Multiply } R_1 \text{ by } A, \\ R_2 \text{ by } B \text{ and } R_3 \text{ by} \\ C. \end{array}$$

$$= \frac{a'^2b'^2c'^2}{ABC} \begin{vmatrix} ABC & -A^2 & A \\ ABC & -B^2 & B \\ ABC & -C^2 & C \end{vmatrix}$$

$$= a'^2b'^2c'^2 \begin{vmatrix} 1 & A & A^2 \\ 1 & B & B^2 \\ 1 & C & C^2 \end{vmatrix} \begin{array}{l} = a'^2b'^2c'^2 (A-B)(B-C) \\ \times (C-A) \text{ (Ex. 1 P 19)} \end{array}$$

$$= a'^2b'^2c'^2 \left(\frac{a}{a'} - \frac{b}{b'}\right)\left(\frac{b}{b'} - \frac{c}{c'}\right)\left(\frac{c}{c'} - \frac{a}{a'}\right)$$

$$= (ab' - a'b)(bc' - b'c)(ca' - c'a). \quad \textbf{Hence Proved.}$$

Q. 7. Prove $\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix}$

$= 1 + a_1 + a_2 + a_3 + a_4$ (Delhi 65 ; Alld. 59 ; Agra 53)

Applying $C_1 + C_2 + C_3 + C_4$ and taking out $1 + a_1 + a_2 + a_3 + a_4$ common from C_1 of new determinant, we get

$$\Delta = (1 + a_1 + a_2 + a_3 + a_4) \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ 1 & 1+a_2 & a_3 & a_4 \\ 1 & a_2 & 1+a_3 & a_4 \\ 1 & a_2 & a_3 & 1+a_4 \end{vmatrix}$$

Now apply R_2-R_1, R_3-R_1 and R_4-R_1.

$$\therefore \quad \Delta=(1+a_1+a_2+a_3+a_4)\begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=1+a_1+a_2+a_3+a_4$$

Q. 8. $\begin{vmatrix} 1+a_1 & a_2 & a_3 \ldots\ldots\ldots & a_n \\ a_1 & 1+a_2 & a_3 \ldots\ldots & a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 \ldots & 1+a_n \end{vmatrix}=1+a_1+a_2+\ldots a_n$

(Raj. 60 ; Agra M. Sc. 66)

Just as Q. 7.

Q. 9. (a) Prove $\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}=x^3(x+10)$ Proceed as above

(Agra 39 ; Patna 41 ; Mysore 42 ; Luck. 49)

(b) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}=(x+3a)(x-a)^3$

(Agra 65 ; Bihar 66 ; Luck. 44 ; Supp., Dehli 47 ; Nagpur 36)

Praceed exactly as in Q 7.

Q. 10. (a) Find the value of determinant of nth order :—

$$\begin{vmatrix} x & a & a & \dots & \dots & a \\ a & x & a & \dots & \dots & a \\ a & a & x & \dots & \dots & a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & \dots & x \end{vmatrix}$$

whose leading constituents are equal to x and the remaining constituents are equal to a.

(Agra M. Sc. 59 ; Raj. 61 ; I. A. S. 53 ; Cal. Hon's 61)

Proceeding exactly as in Q 7, we get

$$\triangle=[x+(n-1)\,a]\,(x-a)^{n-1}.$$

(b) Find the value of the determinant of order n.

$$\begin{vmatrix} 1+x & 1-x & 1-x & \dots & \dots & 1-x \\ 1-x & 1+x & 1-x & \dots & \dots & 1-x \\ 1-x & 1-x & 1+x & \dots & \dots & 1-x \\ \dots\dots & \dots\dots & \dots\dots & \dots & \dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots & \dots & \dots\dots \\ 1-x & 1-x & 1-x & \dots & \dots & 1+x \end{vmatrix}$$

(Gauhati Hons. 64 ; Raj. M. Sc. 59)

Proceeding exactly as in Q. 7, we get

$$\varDelta=[1+x+(n-1)\,(1-x)]\,[(1+x)-(1-x)]^{n-1}$$
$$=[n-(n-2)x]\,(2x)^{n-1}.$$

(c) Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1+k & 1 & \dots & \dots & 1 \\ 1 & 1 & 1+k & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & \dots & 1+k \end{vmatrix}$$

a determinant of nth order in which every constituent of the leading diagonal except the first is (k+1) and all other constituents are unity.

Show also that if all the constituents of leading diagonal are equal to (k+1) and all other unity, the value of determinant is $k^{n-1}(n+k)$. **(Lucknow Prel. 61)**

Apply C_2-C_1, C_3-C_1,...C_n-C_1, you will find that everywhere there will be zero except the first column in which each element is unity and diagonal in which there will be 1, K, K,...K.

$$\therefore \Delta \begin{vmatrix} 1 & 0 & 0 & \dots & \dots & \\ 1 & K & 0 & \dots & \dots & 0 \\ 1 & 0 & K & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & \dots & K \end{vmatrix}$$

Expanding with first row $\Delta=1.K^{n-1}=K^{n-1}$. In the 2nd case if the top corner constituent be also $1+K$, then adding

all the columns to first column we will have a common factor $(n+k)$ which when taken out will leave the above determinant whose value we have shown to be k^{n-1}. Hence the 2nd determinant is $(n+k)\,k^{n-1}$.

(d) Find x if

$$\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix}=0.$$

(Cal. Hons. 65 ; Delhi 53 ; Bombay 53 ; Lucknow Pass 61)

The above determinant can be easily shown to be

$$(x+a+b+c+d)\,x^3=0.$$

$\therefore\quad x=0;\ \text{or}\ -(a+b+c+d).$

11. Prove $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(Agra M. Sc. 54, 64 ; I. A. S. 14 ; Calcutta 48 ; Sagar 66 ; Pb. 48, 57, 58 ; Agra B. Sc. 53, 59, 64, 66 ; Delhi 48, 52)

Dividing R_1, R_2, R_3 and R_4 by a, b, c and d respectively,

$$\Delta=abcd\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

Now apply $R_1+R_2+R_3+R_4$ and proceed as in Q. 7.

Q. 12. (a) Prove

$$\Delta=\begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix}=\begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

$=1+a^2+b^2+c^2+d^2.$ **(Allahabad 49)**

Multiplying C_1, C_2, C_3 and C_4 by a, b, c and d respectively and hence dividing by $abcd$,

$$\Delta=\frac{1}{abcd}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 & ad^2 \\ a^2b & b(b^2+1) & bc^2 & bd^2 \\ b^2c & b^2c & c(c^2+1) & cd^2 \\ a^2d & b^2d & c^2d & d(d^2+1) \end{vmatrix}$$

Now a, b, c and d can be taken out from R_1, R_1, R_3 and R_4 respectively.

$$\therefore\quad \Delta=\frac{abcd}{abcd}\begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

Now proceed as in Q. 7, *i.e.* $C_1+C_2+C_3+C_4$ etc.

(b) Show that

$$\Delta=\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

is divisible by λ^3 and find the other factor.

(Raj. 49 ; Agra M. Sc. 65)

Proceeding exactly as in Q. 12 (a) above by multiplying C_1, C_2, C_3, C_4 by a, b, c, d respectively and taking out a, b, c, d common from R_1, R_2, R_3 and R_4 respectively, we find

$$\Delta=(\lambda+a^2+b^2+c^2+d^2)\,\lambda^3.$$

Hence it is divisible by λ^3 and the other factor is

$$\lambda+a^2+b^2+c^2+d^2.$$

Q. 13. (a) Prove $\begin{vmatrix} x^3 & 3x^2 & 3x & 1 \\ x^2 & x^2+2x & 2x+1 & 1 \\ x & 2x+1 & x+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}=(x-1)^6$

(Venkateswra 66; Jiwaji 66; Agra 41; Luck. 49; Kerala 65)

Applying $C_1-C_2+C_3-C_4$, we get

$$\Delta=(x-1)^3\begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Now make two zeros in C_3 by applying R_1-R_2 and R_2-R_3 and expand by it.

(b) Prove that

$$\begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ 3\alpha^2 & 2\alpha & 1 & 0 \\ \beta^3 & \beta^2 & \beta & 1 \\ 3\beta^2 & 2\beta & 1 & 0 \end{vmatrix}=(\alpha-\beta)^4$$

Apply R_1-R_3 aud R_2-R_4 and take out $(\alpha-\beta)$ commes from each R_1 and R_2

$$\therefore \quad \Delta=(\alpha-\beta)^2 \begin{vmatrix} \alpha^2+\beta^2+\alpha\beta & \alpha+\beta & 1 & 0 \\ 3(\alpha+\beta) & 2 & 0 & 0 \\ \beta^3 & \beta^2 & \beta & 1 \\ 3\beta^2 & 2\beta & 1 & 0 \end{vmatrix}$$

Expanding with 4th column and hence $-$ive sign,

$$\Delta=-(\alpha-\beta)^2 \begin{vmatrix} \alpha^2+\beta^2+\alpha\beta & \alpha+\beta & 1 \\ 3(\alpha+\beta) & 2 & 0 \\ 3\beta^2 & 2\beta & 1 \end{vmatrix}.$$

Now apply R_1-R_3 and again take $(\alpha-\beta)$ out and expand etc.

$$\Delta=(\alpha-\beta)^4.$$

Q. 14. Show that

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}=(bc+ca+ab)^3.$$

(Bombay 61)

Multiply R_1, R_2, R_3 by a, b and c respectively and hence dividing Δ by abc,

$$\Delta=\frac{1}{abc}\begin{vmatrix} -abc & ab(b+c) & ac(c+b) \\ ab(a+c) & -abc & bc(c+a) \\ ac(a+b) & bc(b+a) & -abc \end{vmatrix}$$

Taking a, b, c common from C_1, C_2 and C_3,

$$\Delta=\frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & ac+ab \\ ac+bc & bc+ca & -ab \end{vmatrix}.$$

Applying $R_1+R_2+R_3$ and taking $ab+bc+ca$ common, we get

$$\Delta=(ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ca & -ab \end{vmatrix}$$

Applying C_2-C_1 and C_3-C_1, we get

$$\Delta=(ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ca) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix}$$

$$=(ab+bc+ca)\,(ab+bc+ca)^2=(ab+bc+ca)^3.$$

Q. 15. Prove that

$$\begin{vmatrix} x & a_1 & a_2 & a_3 & 1 \\ \alpha & x & b_1 & b_2 & 1 \\ \alpha & \beta & x & c_1 & 1 \\ \alpha & \beta & \gamma & x & 1 \\ \alpha & \beta & \gamma & \delta & 1 \end{vmatrix}=(x-\alpha)\,(x-\beta)\,(x-\gamma)\,(x-\delta).$$

(Type Mysore

Apply $C_1-\alpha C_5$, $C_2-\beta C_5$, $C_3-\gamma C_5$, $C_4-\delta C_5$.

$$\therefore\ \Delta=\begin{vmatrix} x-\alpha & a_1-\beta & a_2-\gamma & a_3-\delta & 1 \\ 0 & x-\beta & b_1-\gamma & b_2-\delta & 1 \\ 0 & 0 & x-\gamma & c_1-\delta & 1 \\ 0 & 0 & 0 & x-\delta & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

The above is clearly equal to $(x-\alpha)\,(x-\beta)\,(x-\gamma)\,(x-$

*Q. 16. (a) Prove that

$$\Delta=\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}=2abc\,(a+b+c)^3$$

(Gorakhpur 59 ; Cal. 31 ; Agra 50, 62, 65 ; Raj. 51 ; Sagar 63 ; Aligarh B Sc. Hon's 59 ; Agra M.Sc. 56, 58 ; Bombay 58 ; Nagpur 61 ; Lucknow 64 ; Delhi 52).

Applying C_2-C_1, C_3-C_1, we get

$$\Delta=\begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$=\begin{vmatrix} (b+c)^2 & (a+b+c)\,(a-b-c) & (a+b+c)\,(a-b-c) \\ b^2 & (c+a+b)\,(c+a-b) & (a+b+c).0 \\ c^2 & (a+b+c)\,0 & (a+b+c)\,(a+b-c) \end{vmatrix}$$

Taking out $(a+b+c)$ common from C_2 and C_3,

$$\Delta=(a+b+c)^2\begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Now applying $R_1-R_2-R_3$, we get

$$\Delta=(a+b+c)^2\begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $C_2+\frac{1}{b}\,C_1$, $C_3+\frac{1}{c}\,C_1$.

$$\Delta=(a+b+c)^2\begin{vmatrix} 2bc & 0 & 0 \\ b^2 & c+a & b^2/c \\ c^2 & c^2/b & a+b \end{vmatrix}$$

$$=2bc\ (a+b+c)^2\begin{vmatrix} c+a & b^2/c \\ c^2/b & a+b \end{vmatrix}$$

$$=2bc\ (a+b+c)^2\left[(a+b)\ (c+a)-\frac{c^2}{b}\cdot\frac{b^2}{c}\right]$$

$$=2bc\ (a+b+c)^2\ a\ (a+b+c)$$

$$=2abc\ (a+b+c)^3.$$

(b) If $2s=a+b+c$, prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}=2s^3\ (s-a)(s-b)(s-c$$

(I. A. S. 56 ; Cal. 5

Let $s-a=A$, $s-b=B$, $s-c=C$.

$\therefore\ \ A+B+C=3s-(a+b+c)=s.$

$B+C=2s-b-c=a$; similarly $C+A=b$ and $A+B=$

$$\therefore \Delta=\begin{vmatrix} (B+C)^2 & A^2 & A^2 \\ B^2 & (C+A)^2 & B^2 \\ C^2 & C^2 & (A+B)^2 \end{vmatrix}=2ABC\ (A+B+C)^3 \text{ by part (a)}$$

$$=2(s-a)(s-b)(s-c)s^3$$

Q. 17. Prove that

$$\Delta=\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix}=2abc\ (a+b+c)^3.$$

Multiplying R_1, R_2 and R_3 by c, a and b respectively and hence dividing the determinant by abc,

$$\Delta=\frac{1}{abc}\begin{vmatrix} c(a+b)^2 & c^2a & c^2b \\ a^2c & a(b+c)^2 & a^2b \\ b^2c & b^2a & b(c+a)^2 \end{vmatrix}$$

Take c, a and b common from C_1, C_2 and C_3 respectively

$$\therefore\ \Delta=\frac{abc}{abc}\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}.$$

which is of the same form as Q. 16 (a).

+Q. 18. **Prove that**

$$\Delta=\begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix}=(x-y)(y-z)(z-x)\times(x+y+z)(x^2+y^2+z^2)$$

[Agra 45, 56 ; Alld. 55, 50, 32 ; Vikram 61 ; Nagpur 54 (S) ; Celhi Hons. 60 ; Ph 57 (S)]

Split the given determinant into two, one of which shall vanish because of identical lines. We get

$$\Delta=\begin{vmatrix} x^2 & -(y^2+z^2)+2yz & yz \\ y^2 & -(z^2+x^2)+2zx & zx \\ z^2 & -(x^2+y^2)+2xy & xy \end{vmatrix}.$$

Again split into two determinants one of which shall vanish.

$$\therefore\ \Delta=-\begin{vmatrix} x^2 & y^2+z^2 & yz \\ y^2 & z^2+x^2 & zx \\ z^2 & x^2+y^2 & xy \end{vmatrix}.$$

Apply C_1+C_2 and take out $x^2+y^2+z^2$.

$$\Delta=-(x^2+y^2+z^2)\begin{vmatrix} 1 & y^2+z^2 & yz \\ 1 & z^2+x^2 & zx \\ 1 & x^2+y^2 & xy \end{vmatrix}.$$

Now apply R_2-R_1, R_3-R_1.

$$\therefore \quad \Delta=-(x^2+y^2+z^2)\begin{vmatrix} 1 & y^2+z^2 & yz \\ 0 & x^2-y^2 & z(x-y) \\ 0 & x^2-z^2 & y(x-z) \end{vmatrix}$$

Expanding with first column and taking $x-y, x-z$ common from R_2 and R_3,

$$\Delta=-(x^2+y^2+z^2)(x-y)(x-z)\begin{vmatrix} x+y & z \\ x+z & y \end{vmatrix}$$

Apply C_1+C_2 and take $x+y+z$ out.

$$\therefore \quad \Delta=(x^2+y^2+z^2)(x-y)(z-x)(x+y+z)\begin{vmatrix} 1 & z \\ 1 & y \end{vmatrix}$$

$$=(x^2+y^2+z^2)(x-y)(y-z)(z-x)(x+y+z).$$

Q. 19. Prove

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ -bc+ca+ab & bc-ca+ab & bc+ca-ab \\ (a+b)(b+c) & (b+c)(b+a) & (c+a)(c+b) \end{vmatrix}$$

$$=3(b-c)(c-a)(a-b)(a+b+c)(bc+ca+ab).$$

(Cal Hons. 52)

Apply $R_1-R_2+R_3$ and taking factor 3 out,

$$\Delta=3\begin{vmatrix} bc & ca & ab \\ -bc+ca+ab & bc-ca+ab & bc+ca-ab \\ a^2+\Sigma ab & b^2+\Sigma ab & c^2+\Sigma ab \end{vmatrix}$$

Apply R_2+2R_1, R_3-R_1, $\Sigma ab=ab+bc+ca$.

$$\Delta=3\begin{vmatrix} ac & ca & ab \\ ab+bc+ca & ab+bc+ca & ab+bc+ca \\ a(a+b+c) & b(a+b+c) & c(a+b+c) \end{vmatrix}$$

Taking $ab+bc+ca$ and $a+b+c$ common from R_2 and R_3.

$$\Delta=3(a+b+c)(ab+bc+ca)\begin{vmatrix} bc & ca & ab \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

Multiply C_1, C_2, C_3 by a, b, c respectively and as such divide Δ by abc. Then take out abc from first row of new Δ. Refer Q. 36 P. 47.

$$\Delta=3(a+b+c)(ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$=3(a+b+c)(ab+bc+ca)(a-b)(b-c)(c-a)$

[Ex. 1 P. 19].

Q. 20. Multiply $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$

and show that product of two expressions of the form $L^3+M^3+N^3-3LMN$ is of the same form.

(I. A. S. 57 ; Gauhati Hons. 64)

$$\text{Product}=\begin{vmatrix} ax+by+cz & ay+bz+cx & az+bx+cy \\ bx+cy+az & by+cz+ax & bz+cx+ay \\ cx+ay+bz & cy+az+bx & cz+ax+' \end{vmatrix}.$$

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

If we multiply in the above form ; the 1st constituent of the 1st row of product determinant will be $a^2+b^2+c^2$, where as we want it to be $2bc-a^2$.

re-arranging the terms of 2nd determinant, we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times - \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

But $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (3abc-a^3-b^3-c^3)^2$
$= (a^3+b^3+c^3-3abc)^2$

By expansion

Q. 22. If $ax_1^2+by_1^2+cz_1^2=d=ax_2^2+by_2^2+cz_2^2$
$=ax_3^2+by_3^2+cz_3^2$ and $ax_1x_2+by_1y_2+cz_1z_2=f$
$=ax_2x_3+by_2y_3+cz_2z_3=ax_1x_3+by_1y_3+cz_1z_3$,

then prove

$$abc \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = (d-f)^2\,(d+2f).$$

(Pb. 33)

$$\text{L.H.S.}=abc\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\times\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$=\begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix}\times\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

On multiplying,

$$=\begin{vmatrix} \Sigma ax_1^2 & \Sigma ax_1x_2 & \Sigma ax_1x_3 \\ \Sigma ax_1x_2 & \Sigma ax_2^2 & \Sigma ax_2x_3 \\ \Sigma ax_1x_3 & \Sigma ax_2x_3 & \Sigma ax_3^2 \end{vmatrix}=\begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$

Now proceed as in question 7 page 74.

Q. 23. Fxpress $\Delta=\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$

as product of two determinants and find its value.

(Agra 32, 37, 49, 51 ; Andhra 92 ; I.A.S. 40 ;
Nagpur 51, 54, 61 ; Allahabead 56 ; I.AS. 55 ;
Calentta Hons. 61)

If we put $x=y$ in Δ, then $C_1=C_2$; $\therefore \quad \Delta=0$.
Hence $x-y$ is a factor.
Similarly $(y-z)$, $(z-x)$ are also factors of Δ.

But $(x-y)(y-z)(z-x)=\begin{vmatrix} 1 & 1 & 1 \\ x & y & y \\ x^2 & y^2 & z^2 \end{vmatrix}$ (Ex. 1 Art. 17 page 19)

Again if we put $a=b$ in, Δ then $R_1=R_2$; $\therefore\ \ \Delta=0$.

Hence $(a-b)$ is a factor.

Similarly $(b-c)$, $(c-a)$ are also factors of Δ. But

$$(a-b)(b-c)(c-c)=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad \text{[Ex. 1 Art. 17 P. 19].}$$

$$\therefore\ \ \Delta=K\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}\times\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

for some such suitable value of K so that on multiplying the two determinants the resulting determinant is Δ.

$$\therefore\ \Delta=K\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}\times\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \qquad \ldots(A)$$

$$=-K\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}\times\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

we have interchanged two columns.

If we multiply we get Ist constituent of product determinant as x^2+ax+a^2 whereas we want it to be $x^2-2ax+a^2$.

$\therefore$ K must be chosen$=2$.

$$\therefore\ \ \Delta=\begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix}\times\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

is required form, because on multiplication it becomes equal to Δ.

The value of $\Delta = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= 2\,(x-y)\,(y-z)\,(z-x)\,(a-b)\,(b-c)\,(c-a).$$

Q. 25. Express (Agra 35)

$$\Delta = \begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$$

as product of two determinants and find its value.

Proceeding as in Q. 24 and taking $\Delta = (A)$ of Q. 24 when $K=2$, its value is $2\,(x-y)\,(y-z)\,(z-x)\,(a-b)\,(b-c)\,(c-a)$

Q. 25. Find the condition in determinant form, in order that the general expression of the second degree in two variables f(x, y)=ax²+2hxy+by²+2gx+2fy+c may be resolved into two linear factors. (Raj. M. Sc. 59)

Let the two linear factors be

$$l_1x+m_1y+n_1 \text{ and } l_2x+m_2y+n_2,$$

Then $f(x, y) = (l_1x+m_1y+n_1)\,(l_2x+m_2y+n_2)$.

Comparing the coefficients,

$$l_1l_2 = a,\ m_1m_2 = b,\ n_1n_2 = c.$$

$$l_1m_2+l_2m_1 = 2h,\ m_1n_2+m_2n_1 = 2f,\ n_1l_2+n_2l_1 = 2g.$$

Now we know that

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0.$$

Multiplying these two zero determinants, we get

$$\begin{vmatrix} 2l_1l_2 & l_1m_2+l_2m_1 & n_1l_2+n_2l_1 \\ m_1l_2+m_2l_1 & 2m_1m_2 & m_1n_2+m_2n_1 \\ n_1l_2+n_2l_1 & m_1n_2+m_2n_1 & 2n_1n_2 \end{vmatrix}=0$$

or
$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix}=0$$

or
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$ is the required condition.

On expanding, it becomes

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

i.e. the condition that general equation of 2nd degree may represent two straight lines.

Q. 26. (a) Express as the product of two determinants and show that its value is zero.

$$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\beta-\gamma) & 1 \end{vmatrix}.$$ **(Punjab 62)**

$$\Delta=\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}\times\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix},$$

each of which is zero. Hence the given determinant

(h) Prove that

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_2\alpha_1+b_2\beta_1 & a_3\alpha_1+b_3\beta_1 \\ a_1\alpha_2+b_1\beta_2 & a_2\alpha_2+b_2\beta_2 & a_3\alpha_2+b_3\beta_2 \\ a_1\alpha_3+b_1\beta_3 & a_1\alpha_3+b_2\beta_3 & a_3\alpha_3+b_3\beta_3 \end{vmatrix}=0.$$

Above is $\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix}$,

each of which is zero. Hence the given determinant is zero.

(c) Prose that

$$\Delta=\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}=0$$

We know that

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & \gamma+\delta & \alpha+\beta \\ 0 & \gamma\delta & \alpha\beta \end{vmatrix}=0 \text{ and } \begin{vmatrix} 0 & 1 & 1 \\ 0 & \alpha+\beta & \gamma+\delta \\ 0 & \alpha\beta & \gamma\delta \end{vmatrix}=0.$$

On multiplying these two zero determinants, wet get Δ which therefore is also equal to zero.

***Q. 17. Prove that**

$$\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & c^2+a^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix}=(bc+ca+ab)^2.$$

(Raj. 50)

By inspection, we find that

$$\Delta=\begin{vmatrix} 0 & b & c & 1 \\ a & 0 & c & 1 \\ a & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}\times\begin{vmatrix} 0 & b & c & 1 \\ a & 0 & c & 1 \\ a & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=\Delta'^2$$

where Δ' is easily seen by applying C_2-bC_4, C_3-cC_4 and expanding, to be equal to $(bc+ca+ab)$.

$$\Delta'=\begin{vmatrix} 0 & 0 & 0 & 1 \\ a & -b & 0 & 1 \\ a & 0 & -c & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=-\begin{vmatrix} a & -b & 0 \\ a & 0 & -c \\ 1 & 1 & 1 \end{vmatrix}$$

$$=-[a\,(0+c)+b\,(a+c)]=-(ab+bc+ca).$$

$$\therefore\quad \Delta^2=(ab+bc+ca)^2.$$

***Q. 28. Show that**

$$\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix}=\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$$

Let the determinant on L.H.S. be $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Cofactor of x is $a^2+x^2=A_1$.

Cofactor c is $-(-cx-ab)=ab+cx=B_1$

Cofactor of $-b$ is $ac-bx=C_1$.

$$\therefore \quad \text{L.H.S.} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

by Q. 47 P. 5

CHAPTER II

DETERMINANTS OF SPECIAL TYPE

§ 21. Second Minor.

We have already defined in § 11 as to what we mean by the minor of any particular constituent in a determinant *i.e.* the determinant which is left by suppressing the rows and columns that intersect at that particular constituent. Now we shall call this minor as first minor showing that one row and one column intersecting at that particular constituent have been suppressed. The determinant that is left is called minor determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

If we now suppress two rows and two columns then the determinant that is left is called second minor *i.e.* if we suppress the first two rows and first two columns then the determinant left is $\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$ *i e.* $c_3d_4 - c_4d_3$ which is called second minor.

Complementary to 2nd minor. We observe that the suppressed rows and columns have common constituents

which form a determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and this determinant

is called complementary of the second minor $\begin{vmatrix} c_3 & d_3 \\ c_4' & d_4 \end{vmatrix}$.

Notation. The first minor of a determinant Δ is denoted by Δa_1, *i.e.* the determinant which is left by suppressing the rows and columns intersecting at a_1. It is also denoted by the corresponding capital letter A_1.

Similarly the second minor of a determinant is denoted by $\Delta a_1 b_2$, *i.e.* the determinant which is left by suppressing the rows and columns which enclose a determinant whose leading term is $a_1 b_2$ and so on.

§ 22. Expansion of a determinant in terms of minors of any order. Laplace's Method.

(M. Sc. 56, 50 ; Nagpur 61)

We have so far expanded a determinant in terms of constituents of one row and one column. Now we shall expand the determinant in terms of two rows or two columns.

Rule :— In terms of first two rows. Make all possible determinants from these two rows by taking any two columns of these, *i.e.* there are four columns and by taking two columns at a time, we shall have six determinants of the type

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}.$$

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}, \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}.$$

Multiply each of them by the corresponding determinant which is left by suppressing the rows and columns intersecting at them, *i.e.* by their minors and then add them with proper signs, the rule for which is as follows.

Just as while expanding in terms of one row and one column we take alternately +ive and −ive, here we shall see that how many movements of rows or columns will make each of the above rows and columns the first two rows or columns. If it is odd, then sign to be attached is −ve and if even, then +ive ;

e.g.

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Now the c column being 3rd can be made 2nd by one movement of columns ; a column is in the position of first column so that the total number of movements is one, *i.e.* odd ; hence the sign will be −ive.

Similarly

$$\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}.$$

Now b column being 2nd can be made first by one movement and d column being 4th can be made 2nd by two movements and hence the total movements are $2+1=3$, *i.e.* odd ; therefore minus sign.

Similarly

$$\begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}.$$

Now c column being 3rd can be made first by two movements and d column being fourth can be made 2nd by two movements.

Hence the total movements are $2+2=4$, *i.e.* even; therefore plus sign.

Therefore in the light of above the expansion of the determinant is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix}$$

$$+\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix}+\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix}$$

$$-\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}\begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix}+\begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}$$

The above expansion is known as Laplace's expansion of a determinant.

The method will be of great use when there are zeros in any two rows or two columns, *e.g.*

$$\Delta=\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

By the above rule the determinant will be

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}-\begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix}\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} \text{ i.e. zero}$$

$$+\begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix}\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} \text{ i.e. zero}+\begin{vmatrix} b_1 & 0 \\ b_2 & 0 \end{vmatrix}\begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} \text{ i.e. zero}$$

$$-\begin{vmatrix} b_1 & 0 \\ b_2 & 0 \end{vmatrix}\begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} \text{ i.e. zero}-\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \text{ i.e. zero}$$

$$=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$$

In a similar manner we can prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & p_1 & q_1 & r_1 \\ a_2 & b_2 & c_2 & p_2 & q_2 & r_2 \\ a_3 & b_3 & c_3 & p_3 & q_3 & r_3 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Because if we expand the determinant by the help of first three columns and their minors, we shall get only product given in the R. H. S. All other determinants shall vanish because one row will be zero.

From above we conclude that if a determinant of 2mth order contains in any position a square of m^2 zeroes it can be expressed as the product of two determinants of the mth order. **(Agra M. Sc. 50)**

§·23. Reciprocal Determinant.

Definition :—If in a given determinant each constituent be replaced by its **cofactor**, then the determinant so formed is called reciprocal or inverse determinant. If the original determinant be Δ, then we may denote the reciprocal determinant by Δ'.

Properties :—$\Delta' = \Delta^{n-1}$ where n is the order of the determinant i.e. the reciprocal determinant=(original determinant)$^{n-1}$.

(Cal. Hons. 65; Agra M. Sc. 64, 66; Delhi Hons. 65)

We have already proved the above property in the case of a determinant of 3rd order, i.e.

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

[Q. 47, P. 52]

i.e. $\Delta' = \Delta^2 = \Delta^{3-1}$.

Similarly we may consider a determinant of the nth order.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & \dots & n_1 \\ a_2 & b_2 & c_2 & \dots & n_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & n_n \end{vmatrix}$$

$$\therefore \quad \Delta' = \begin{vmatrix} A_1 & B_1 & C_1 & \dots & N_1 \\ A_2 & B_2 & C_2 & \dots & N_2 \\ \dots & \dots & \dots & \dots & \dots \\ A_n & B_n & C_n & \dots & N_n \end{vmatrix}$$

Also we know that

$$a_1A_1 + b_1B_1 + c_1C_1 + \dots n_1N_1 = \Delta \text{ etc.}$$

and $$a_1A_2 + b_1B_2 + c_1C_2 + \dots nN_2 = 0 \text{ etc.}$$ [§ 12 P. 12–13]

With the help of above we have

$$\Delta\times\Delta'=\begin{vmatrix} \Delta & 0 & 0 & \dots & 0 \\ 0 & \Delta & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \Delta \end{vmatrix}=\Delta^n.$$

$$\therefore\quad \Delta'=\Delta^{n-1}$$

Hence proved.

Cor. *The product of two reciprocal determinants is the reciprocal determinant of the product of the two orignal determinants.*

If Δ_1 and Δ_2 be the original determinants then their product is $\Delta_1\Delta_2$ which is itself a determinant of nth order and its reciprocal is

$$(\Delta_1\Delta_2)^{n-1}=\Delta_1^{\,n-1}\Delta_2^{\,n-1}$$

But by property 1, Δ_1^{n-1} is reciprocal determinant of Δ_1 and Δ_2^{n-1} is reciprocal determinant of Δ_2 and hence product of two reciprocal determinants.

Property 2. *To find the minor of a reciprocal determinant Δ' in terms of original determinant Δ.* **(Karnatak 62 ; Punjab 60)**

Let us first consider a determinant of fourth order say

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \text{ and } \Delta'=\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

Δ'_{A_1}=first minor of Δ' which is of order 3.

$$=\begin{vmatrix} B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \\ B_4 & C_4 & D_4 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}. \qquad \textbf{(Note carefully)}$$

$$\therefore \quad \Delta \times \Delta'_{A_1}=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & \Delta & 0 & 0 \\ a_3 & 0 & \Delta & 0 \\ a_4 & 0 & 0 & \Delta \end{vmatrix}=a_1\Delta^3.$$

$$\therefore \quad \Delta'_{A_1}=a_1\,\frac{\Delta^3}{\Delta}=a_1\Delta^2=a_1\Delta^{3-1}. \quad ..(1)$$

Now complementary of the minor Δ'_{A_1} of 3rd order is A_1 and the corresponding from the original determinant Δ is a_1.

Again let us consider a minor of 2nd order of reciprocal determinant.

$\Delta'_{A_1B_2}$ = 2nd minor of Δ' which is of order 2. It is obtained by suppressing the rows and columns that constitute the determinant whose leading terms is A_1B_2.

$$\because \quad \Delta'_{A_1B_2} = \begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ B_3 & C_3 & D_3 \\ B_4 & C_4 & D_4 \end{vmatrix} \quad \text{(Note carefully)}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}. \quad \text{(Note carefully)}$$

$$\therefore \quad \Delta \times \Delta'_{A_1B_2}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ A_3 & B_3 & C_3 & D_4 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & \Delta & 0 \\ a_4 & b_4 & 0 & \Delta \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} \Delta & 0 \\ 0 & \Delta \end{vmatrix}$$

+all other determinants vanish (Laplace method).

We have expanded in terms of minors of 2nd order formed from the first two rows.

$$\therefore \quad \Delta \times \Delta'_{A_1B_2} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \Delta^2 + \text{all other determinants vanishing.}$$

$$\Delta'_{A_1B_2} = (a_1b_2 - a_2b_1) \frac{\Delta^2}{\Delta} = (a_1b_2 - a_2b_1)\, \Delta^{2-1}.$$

Now complementary of the minor $\Delta'_{A_1 B_2}$ of 2nd order is $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$ and the corresponding from the original determinant is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$.

Similarly if we have a determinants of 5th order, then

$$\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \\ A_4 & B_4 & C_4 & D_4 & E_4 \\ A_5 & B_5 & C_5 & D_5 & E_5 \end{vmatrix}.$$

Then $\Delta'_{A_1 B_2} = \begin{vmatrix} C_3 & D_3 & E_3 \\ C_4 & D_4 & E_4 \\ C_5 & D_5 & E_5 \end{vmatrix}$.

i e. minor of 3rd order and it can be shown to be

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \Delta^{3-1} = (a_1 b_2 - a_2 b_1)\, \Delta^2.$$

In the light of above we have the general property, that a minor of order p of the reciprocal determinant is equal to the complementary of that minor (but taken from the original determinant) multiplied by Δ^{p-1}, i e. (original determinant)$^{p-1}$

Property 3. *If a determinant vanishes then its reciprocal determinant also vanishes*

$\Delta' = \Delta^{n-1}$ (property 1) $= 0$, $\because$ $\Delta = 0$ (given)

Property 4. *If a determinant vanishes then each minor of the reciprocal determinant vanishes. and the constituent of any*

row of the reciprocal determinant are proportional to those of other row and similarly the columns

By property 2 each minor of order p will contain Δ^{p-1} as its factor and as such it is zero because $\Delta=0$ (given).

Let us consider a determinant of fourth order.

$$\Delta' A_1B_2=\begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix}=(a_1b_2-a_2b_1)\,\Delta^{2-1}=0, \quad \therefore \quad \Delta=0.$$

$$\therefore \quad C_3D_4-C_4D_3=0 \quad \text{or} \quad \frac{C_3}{C_4}=\frac{D_3}{D_4} \qquad ...(1)$$

Similarly $\Delta' B_1C_2=\begin{vmatrix} A_3 & D_3 \\ A_4 & D_4 \end{vmatrix}=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\Delta^{2-1}=0 \because \Delta=0$

$$\therefore \quad A_3D_4-A_4D_3=0, \text{ i.e. } \frac{A_3}{A_4}=\frac{D_3}{D_4}=\frac{C_3}{C_4} \text{ from (1).}$$

Similarly we can prove that $\dfrac{A_3}{A_4}=\dfrac{B_3}{B_4}=\dfrac{C_3}{C_4}=\dfrac{D_3}{D_4}$, *i e.* the elements of 3rd row of reciprocal determinant are proportional to those of fourth row or in general to any row. Similarly we may prove for columns.

§ 25. Symmetrical Determinants. (Agra M. Sc. 51)

Conjugate Elements. Any two constituents of a determinant are said to be conjugate if they are situated on the opposite sides of the principal diagonal in a line perpendicular to it and at equal distances from it, *i e.* in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ the elements b_1, a_2; c_1, a_3; d_1, a_4; c_2, b_3; d_2, b_4; d_3, c_4 are all conjugate elements.

If in a determinant every two conjugate elements are equal

then such a determinant is called symmetrical determinant.

e g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is a symmetrical determinant.

Properties. It is easy to judge that the cofactors of an two conjugate elements of a symmetrical determinant ar same, *i e.* cofactor of h in first row is $-(ch-fg)$ and of h i first column is $-(ch-fg)$, *i e.* they are equal. Similarly tha of g in first row is $+(hf-bg)$ and of g in first column i $+(hf-bg)$ *i.e.* they are equal

No 1. *Hence we conclude that reciprocal of a symmetri cal determinant is also a symmetrical determinant.*

Again

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2+b_1^2+c_1^2 & a_1a_2+b_1b_2+c_1c_2 & a_1a_3+b_1b_3+c_1c_3 \\ a_1a_2+b_1b_2+c_1c_2 & a_2^2+b_2^2+c_2^2 & a_2a_3+b_2b_3+c_2c_3 \\ a_1a_3+b_1b_3+c_1c_3 & a_2a_3+b_2b_3+c_2c_3 & a_3^2+b_3^2+c_3^2 \end{vmatrix}$$

which is a symmetrical determinant as every two conjugate elements are equal.

No. 2. *Hence we conclude that square of any determi nant is a symmetrical determinant*

Again consider a symmetrical determinant of 4th order.

i.e. $\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix}$.

First minor of a is $\begin{vmatrix} b & f & q \\ f & c & r \\ q & r & d \end{vmatrix}$ which is symmetrical.

First minor of b is $\begin{vmatrix} a & g & p \\ g & c & r \\ p & r & d \end{vmatrix}$ which is also symmetrical.

Similarly first minors of e ond d are also symmetrical determinants.

No. 3. *Hence we conclude that all the leading first minors of a symmetrical determinant are symmetrical determinants.*

Ortho-symmetrical Determinants. A symmetrical determinant in which all the elements of the principal diagonal are equal is said to be ortho-symmetric determinant,

e.g. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$ is ortho symmetric.

[See Q. 42 P. 49]

In order to factorize the above type of determinants as explained in Q. 42 P. 49, we add all the columns to first and find that $a+b+c+d$ is a factor. Again we add first and 2nd columns and subtract 3rd and 4th, thus giving $a+b-c-d$ as a factor. Again add 1st and 3rd and subtract 2nd and 4th thus giving $a+c-b-d$ as a factor.

Similarly we add first and fourth and subtract 2nd and 3rd columns giving $a+d-b-c$ as a factor. Having found all the factors the remaining factor which will be a constant

is obtained by comparing the coefficient of a^4 of the product of diagonal element with the corresponding coefficient of a^4 in the product of the factors.

(See also Q. 44 P 50)

§25. Skew and Skew-Symmetric Determinants.

(Vikram M. Sc. 59 ; Agra M. Sc. 51, 57, 60)

Skew Determinant. *A determinant in which every element on one side of the principal diagonal is equal to its conjugate* with sign changed *is called skew determinant, e. g.*

$$\begin{vmatrix} a & h & g & p \\ -h & b & f & q \\ -g & -f & c & r \\ -p & -q & -r & d \end{vmatrix}$$ is skew determinant.

Skew- Symmetric Determinant. *A determinant in which every element (including those in the principal diagonal) are equal to their conjugate with sign changed is called a skew-symmetric determinant.* (Nagpur 61)

Since every element in the principal diagonal is its conjugate, therefore by definition of skew-symmetric determinant a should be equal to $-a$ *i.e.* $2a=0$ or $a=0$.

Similarly $b=0$, $c=0$, $d=0$ *i.e.* all the elements in principal diagonal should be zero.

Hence a skew-symmetric determinant is zero axial.

Property 1. *A skew-symmetric determinant of odd order vanishes.* (Nagpur M. Sc. 61 ; Nagpur B. Sc. 61 ; Sankati Hons. 65 ; Calcutta Hons. 53 ; Agra M. Sc. 43, 45, 49, 60 ; Pb. 59)

Let us consider a skew-symmetric determinant of fifth order.

$$\Delta=\begin{vmatrix} 0 & h & g & p & l \\ -h & 0 & f & q & m \\ -g & -f & 0 & r & n \\ -p & -q & -r & 0 & t \\ -l & -m & -n & -t & 0 \end{vmatrix}$$

Now we know that if in a determinant rows be changed to columns and columns to rows then the determinant remains unchanged.

$$\therefore\quad \Delta=\begin{vmatrix} 0 & -h & -g & -p & -l \\ h & 0 & -f & -q & -m \\ g & f & 0 & -r & -n \\ p & q & r & 0 & -t \\ l & m & n & t & 0 \end{vmatrix}$$

Now take out (-1) common from each row.

$$\therefore\quad \Delta=(-1)^5\begin{vmatrix} 0 & h & g & p & l \\ -h & 0 & f & q & m \\ -g & -f & 0 & r & n \\ -p & -q & -r & 0 & t \\ -l & -m & -n & -t & 0 \end{vmatrix}$$

$\therefore\ \Delta=-\Delta$ or $2\Delta=0$ *i.e.* $\Delta=0$. **Hence proved.**

Property 2. *The reciprocal of skew-symmetric determinant of* **odd** *order is a* **symmetric determinant.**

In order to find the reciprocal determinant we have to replace each constituent by its co-factor and in order to prove that it is symmetric we must prove co-factors corresponding to every two conjugate elements are equal and of the same sign.

Consider the skew-symmetric determinant of 5th order given above.

Co-factor corresponding to h in the first row is

$$-\begin{vmatrix} -h & f & q & m \\ -g & 0 & r & n \\ -p & -r & 0 & -t \\ -l & -n & -t & 0 \end{vmatrix}. \qquad ...(A)$$

Co-factor corresponding to $-h$ in the first column which is conjugate of h in the first row is

$$-\begin{vmatrix} h & g & p & l \\ -f & 0 & r & n \\ -q & -r & 0 & t \\ -m & -n & -t & 0 \end{vmatrix} = -\begin{vmatrix} h & -f & -q & -m \\ g & 0 & -r & -n \\ p & r & 0 & -t \\ l & n & t & 0 \end{vmatrix}$$

by changing rows into columns and columns into rows.

Take out (-1) common from each row

$$=-(-1)^4\begin{vmatrix} -h & f & q & m \\ -g & 0 & r & n \\ -p & -r & 0 & t \\ -l & -n & -t & 0 \end{vmatrix} = -\begin{vmatrix} -h & f & q & m \\ -g & 0 & r & n \\ -p & -r & 0 & t \\ -l & -n & -t & 0 \end{vmatrix}$$

which is same as (A) and also with the same sign.

Similarly we can prove that co-factors corresponding to any two conjugate elements are equal with the same sign.

Hence the reciprocal of a skew-symmetric determinant of odd order is a symmetric determinant.

Property 3. *The reciprocal of a skew-symmetic determinant of* even *order is a* skew-symmetric determinant. (Agra M.Sc. 50)

In order to find the reciprocal determinant, we have to replace each element by its co-factor and in order to prove that it is skew-symmetric determinant, we must prove that co-factors corresponding to every two conjugate elements are equal and of opposite signs. Also we should show that co-factors of the terms in the principal diagonal are zero as a skew-symmetric determinant is zero axial.

Let us consider a skew-symmetric determinant of even order say
$$\begin{vmatrix} 0 & h & g & p \\ -h & 0 & f & q \\ -g & -f & 0 & r \\ -p & -q & -r & 0 \end{vmatrix}$$

Co-factor corresponding to h in the first row is
$$-\begin{vmatrix} -h & f & q \\ -g & 0 & r \\ -p & -r & 0 \end{vmatrix} \qquad ...(A)$$

Co-factor corresponding to $-h$ in the first column which is conjugate of h in the first row is
$$-\begin{vmatrix} -h & g & p \\ -f & 0 & r \\ -q & -r & 0 \end{vmatrix} = -\begin{vmatrix} h & -f & -q \\ g & 0 & -r \\ p & r & 0 \end{vmatrix}$$
by changing rows into columns and columns into rows.

Take out (-1) common from each row.
$$= -(-1)^3 \begin{vmatrix} -h & f & q \\ -g & 0 & r \\ -p & -r & 0 \end{vmatrix} = + \begin{vmatrix} -h & f & q \\ -g & 0 & r \\ -p & -r & 0 \end{vmatrix}$$
which is the same as (A) but with opposite sign.

Similarly we can prove that cofactors corresponding to any two conjugate elements are equal but with opposite signs. Again cofactor of the elements of the principal diagonal *e g.* of 0 in the top is
$$\begin{vmatrix} 0 & f & q \\ -f & 0 & r \\ -q & -r & 0 \end{vmatrix}$$
which is a skew-symmetric determinant of odd order any by Property (1) it is zero. Similarly co-factors of all the elements in the principal diagonal are zero. Hence all the above things combined together prove that reciprocal of a skew-symmetric determinant of even order is a skew-symmetric determinant.

Property 4. *A skew-symmetric determinant of even order is a perfect square.* **(Vikram M. Sc. 59 ; Agra M. Sc. 52, 55 (57, 64, 66 ; Cal. Hons's. 64 ; I. A. S. 55 ; Nagpur 61 ; Karuatak 62)**

Let us consider a determinant of fourth order,

$$\Delta = \begin{vmatrix} 0 & h & g & p \\ -h & 0 & f & q \\ -g & -f & 0 & r \\ -p & -q & -r & 0 \end{vmatrix}$$ aud let its reciprocal determinant be

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$ which is a skew-symmetric determinant by Property 3

$\therefore A_1=B_2=C_3=D_4=0$ and every two conjugate elements are equal and of opposite signs, *i.e.* $B_1=-A_2$, $C_1=-A_3$, $D_1=-A_4$, $C_2=-B_3$, $D_2=-B_4$ etc

Now $\Delta_{A_1B_2}=\begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix}=\begin{vmatrix} 0 & h \\ -h & 0 \end{vmatrix}\Delta^{2-1}$ (Prop. 2 P. 104)

or $(C_3D_4-C_4D_3)=h^2\Delta$.

Now $C_3=D_4=0$ and $D_3=-C_4$.

$\therefore \quad C_4^2=h^2\Delta$ or $\Delta=\dfrac{C_4^2}{h^2}=\left(\dfrac{C_4}{h}\right)^2$, *i.e.* perfect square.

Hence proved.

Now C_4 is cofactor of $-r=-\begin{vmatrix} 0 & h & p \\ -h & 0 & q \\ -g & -f & r \end{vmatrix}$

$$=-[h\,(hr+pf)-g\,(hq)]=-h\,[pf-gq+hr].$$

$$\therefore \quad \Delta=\frac{C_4^2}{h^2}=(pf-gq+hr)^2.$$

Rule. Write down the last three elements of first row in one line ; then last two of 2nd row and last of third row in one line, *i.e.*

$$\begin{matrix} h & g & p \\ f & q & r. \end{matrix}$$

The value is written as under.

Add the cross product of extremes and subtract the product of middle and then square, i.e. $(hr-gq+pf)^2$.

If $\Delta=\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$ then write $\begin{matrix} a & b & c \\ d & e & f \end{matrix}$

$\therefore \Delta=(af-be+cd)^2$.

(Agra M. Sc. 48 ; I. A. S. 52)

Note. Alternative method for this may be seen in Q. 27, page 40.

Property 5. *The square of any determinant of even order can be expressed as a skew- symmetric determinant.*

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} -b_1 & a_1 & -d_1 & c_1 \\ -b_2 & a_2 & -d_2 & c_2 \\ -b_3 & a_3 & -d_3 & c_3 \\ -b_4 & a_4 & -d_4 & c_4 \end{vmatrix}$

$\therefore \Delta^2 = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \cdot \begin{vmatrix} -b_1 & a_1 & -d_1 & c_1 \\ -b_2 & a_2 & -d_2 & c_2 \\ -b_3 & a_3 & -d_3 & c_3 \\ -b_4 & a_4 & -d_4 & c_4 \end{vmatrix}$

Now multiply the determinants and you will find it to be of the form

$$\begin{vmatrix} 0 & A & B & C \\ -A & 0 & p & q \\ -B & -p & 0 & r \\ -C & -q & -r & 0 \end{vmatrix},$$

which is a skew-symmetric determinant of even order whose value is $(Ar-Bq+Cp)^2$.

§ 26. *If a determinant Δ whose value is zero be bordered in any manner, the product of the determinant so formed say Δ' by the leading first minor of Δ (i.e. leading 2nd minor of Δ') is equal to the product of two homogeneous functions of added constituents.*

Let us choose a determinant of fourth order.

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=0 \text{ (given).}$$

If capital letters denote the corresponding cofactors, then it can be written as

$$\left.\begin{aligned} \Delta&=a_1A_1+a_2\Delta_2+a_3A_3+a_4A_4 \\ &=a_1A_1+b_1B_1+c_1C_1+d_1D_1 \end{aligned}\right]. \qquad \ldots(1)$$

Its leading first minor is $\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$. ...(2)

Let us border this determinant as shown below :

$$\Delta'=\begin{vmatrix} d_0 & \alpha & \beta & \gamma & \delta \\ \alpha' & a_1 & b_1 & c_1 & d_1 \\ \beta' & a_2 & b_2 & c_2 & d_2 \\ \gamma' & a_3 & b_3 & c_3 & d_3 \\ \delta' & a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Its reciprocal determinant may be written as

$$\begin{vmatrix} k_1 & k_2 & \ldots \\ k_3 & k_4 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

Now,

$\begin{vmatrix} k_1 & k_2 \\ k_3 & k_4 \end{vmatrix}$ is a minor of 2nd order of the reciprocal determinant.

$=\Delta'^{2-1}$ multiplied by complementary of the minor but taken from Δ' (not from reciprocal)

$=\Delta' \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} = \Delta' \times$ leading first minor of Δ (Prop. 2. P. 107).

$\therefore\ \Delta' \times$ leading first minor of $\Delta = k_1 k_4 - k_2 k_3$. ...(3)

Now we have to find the values of k_1, k_2, k_3, k_4 which are cofactors of a_0, α, α' and a_1 of Δ'.

$k_1 =$ cofactor of a_0 of $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$ (given)

Since $k_1 = 0$, we need not calculate k_4.

$k_2 =$ cofactor of α of $\Delta' = -\begin{vmatrix} \alpha' & b_1 & c_1 & d_1 \\ \beta' & b_2 & c_2 & d_2 \\ \gamma' & b_3 & c_3 & d_3 \\ \delta' & b_4 & c_4 & d_4 \end{vmatrix}$

$= -(A_1\alpha' + A_2\beta' + A_3\gamma' + A_4\delta')$ from (1),

$k_3 =$ cofactor of α' of $\Delta' = -\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

$= -(A_1\alpha + B_1\beta + C_1\gamma + D_1\delta)$.

$\therefore$ from (3), we get

$\Delta' \times$ leading first minor of $\Delta = -k_2 k_3$

$$= -(A_1\alpha + B_1\beta + C_1\gamma + D_1\delta)\,(A_1\alpha' + A_2\beta' + A\ \gamma' + A_4\delta') \quad \ldots(4)$$

i.e. product of two linear homogeneous functions of the added constituents α, β, γ, δ and α', β', γ', δ'.

Cor. If the given determinant be symmetrical then its reciprocal determinant is also symmetrical (Prop. P. 107) *i.e.* cofactors of every two conjugate elements are equal. Therefore $B_1 = A_2$, $C_1 = A_3$, $D_1 = A_4$. Now if $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$, $\delta' = \delta$, then the two factors in (4) in the R. H S. become equal *i.e.* a square. Hence we have the following:

If a symmetrical determinant whose value is zero be bordered symmetrically (*i.e.* $\alpha = \alpha'$, $\beta = \beta'$), then the product of the determinant so formed by its **leading 2nd minor** (which will be leading first minor of original given determinant, Note) is equal to square with a —ive sign of linear homogeneous function of the bordering constituents *i.e.* $\Delta' \times$ its leading 2nd minor or $\Delta' \times$ first minor of Δ

$$= -(A_1\alpha + A_2\beta + A_3\gamma + A_4\delta)^2 \quad \ldots(5)$$

Another form. Now Δ' is obtained by bordering a symmetrical determinant Δ symmetrically and as such Δ' itself is symmetrical and its leading first minor *i.e.* Δ is zero (given) and its leading 2nd minor is leading first minor of given determinant Δ.

Therefore regarding Δ' as a given determinant we can say in the light of above that

If in any symmetrical determinant (Δ' is symmetrical) the leading first minor vanishes (i.e. $\Delta = 0$), the determinant itself and its leading 2nd minor (i.e. first minor of Δ) have opposite

$\therefore \quad \Delta' \times$ leading first minor of Δ

$= -(A_1\alpha + B_1\beta + C_1\gamma)\ (A_1\alpha' + A_2\beta' + A_3\gamma')$ (§ **26 P. 121**)

or $\Delta' \times \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = -(a^2\alpha + ab\beta + ac\gamma)\ (a^2\alpha' + ab\beta' + ca\gamma')$.

Cancel a^2 from both sides

$\therefore \quad \Delta' = -(a\alpha + b\beta + c\gamma)\ (a\alpha' + b\beta' + c\gamma')$.

§ **27. Expansion of a determinant in terms of the constituents of principal diagonal.**

Let there be a determinant $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$.

Just as we expand the determinant in terms of any row or any column but here we want to expand in terms of constituents of leading diagonal, *i.e.* a_1, b_2, c_3, d_4. For the sake of distinction and clarity let us take them to be A, B, C and D respectively, so that

$$\Delta = \begin{vmatrix} A & b_1 & c_1 & d_1 \\ a_2 & B & c_2 & d_2 \\ a_3 & b_3 & C & d_3 \\ a_4 & b_4 & c_4 & D \end{vmatrix} \qquad \ldots(1)$$

The expansion of this determinant is written as follows:

$$\Delta = \Delta_0 + \Sigma p.A + \Sigma q.AB + ABCD.$$

(Note that there is no term of the typ: $\Sigma r.ABC$.)

Δ_0 contains those therms of Δ which do not contain any of the diagonal elements and is, therefore, obtained by putting all A, B, C and D equal to zero.

$$\therefore \quad \Delta_0 = \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix} \qquad ...(2)$$

ΣpA contains those terms in which the leading coefficients occur one at a time. Clearly this sigma will contain four terms, one each with A, B, C and D. In order to find the coefficient of A, we put B, C, D equal to zero in the determinant that is left after crossing the rows and columns which intersect at A, thus giving the coefficient of A as

$$\begin{vmatrix} 0 & c_2 & d_2 \\ b_3 & 0 & d_3 \\ b_4 & c_4 & 0 \end{vmatrix}$$

Similarly the coefficient of B is obtained by putting A, C and D equal to zero in the determinant that is left after crossing the row and column intersecting at B.

$$\therefore \quad \text{coefficient of } B \text{ is } \begin{vmatrix} 0 & c_1 & d_1 \\ a_3 & 0 & d_3 \\ a_4 & c_4 & 0 \end{vmatrix}$$

Similarly we can write down the coefficients of C and D. Thus we get all the terms of $\Sigma p.A$.

$\Sigma q.\ AB$ contains those terms in which the leading coefficients occur two at a time. There being four leading coefficients, their combinations taken two at a time will give ${}^4C_2$, *i.e.* 6 terms in the sigma of the type AB, AC, AD, BC, BD and CD.

In order to find the coefficient of AB we put C and D equal to zero in the determinant that is left after crossing

the two rows and two columns that contain A and B. Thus the coefficients of AB is $\begin{vmatrix} 0 & d_3 \\ c_4 & 0 \end{vmatrix}$.

Similarly coefficient of BC is $\begin{vmatrix} 0 & d_1 \\ a_4 & 0 \end{vmatrix}$ etc.

We have seen that three is no terms of the type $\Sigma r.\ ABC$ because for that we will have to put $D=0$ in the determinant that is left after crossing the rows and columns which contain A, B, and C and clearly it is zero *i.e.* coefficient of ABC is zero. Similarly coefficient of all other terms of type ABC, CDA and CDB will be zero when the determinant is of the fourth order. The last terms in the above expansion is clearly $ABCD$.

If however the determinant be of fith order then it will not contain any terms of the type $\Sigma s.ABCD$, *i.e.* which contains four leading coefficients at a time.

$$\Delta=\begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}$$

$$+A\begin{vmatrix} 0 & c_2 & d_2 \\ b_3 & 0 & d_3 \\ b_4 & c_4 & 0 \end{vmatrix}+B\begin{vmatrix} 0 & c_1 & d_1 \\ a_3 & 0 & d_3 \\ a_4 & c_4 & 0 \end{vmatrix}$$

$$+C\begin{vmatrix} 0 & b_1 & d_1 \\ a_2 & 0 & d_2 \\ a_4 & b_4 & 0 \end{vmatrix}+D\begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & b_3 & 0 \end{vmatrix}$$

$$+AB\begin{vmatrix} 0 & d_3 \\ c_4 & 0 \end{vmatrix}+AC\begin{vmatrix} 0 & d_2 \\ b_4 & 0 \end{vmatrix}+DA\begin{vmatrix} 0 & c_2 \\ b_3 & 0 \end{vmatrix}$$

$$+BC\begin{vmatrix} 0 & d_1 \\ a_4 & 0 \end{vmatrix}+BC\begin{vmatrix} 0 & c_1 \\ a_3 & 0 \end{vmatrix}+CD\begin{vmatrix} 0 & b_1 \\ a_2 & 0 \end{vmatrix}$$

$+ABCD.$

Ex. 2. Expand the skew-determinant

$$\begin{vmatrix} x & a & b & c \\ -a & x & d & e \\ -b & -d & x & f \\ -c & -e & -f & x \end{vmatrix}$$ **in powers of x.**

Since we have to expand the determinant in powers of x *i.e.* the constituent of diagonal, we follow the method explained in § 27.

Δ_0=constant term in which no x occurs is obtaind by putting all the x's in diagonal elements equal to zero.

$$=\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$ *which is skew-symmetric* determinant of even order and is therefore a perfect square **(Property 4. P. 116)**

whose value is $(af-be+cd)^2$. (See Rule P. 114)

The coefficient of x^3 *i.e.* $x.x.x$ will be clearly zero.

The coefficient of x^2 *i.e.* $x.x$ in the Ist and 2nd columns and rows is obtained by putting the two x's equal to zero in the determinant which is left by corssing the first and second rows

and columns and is therefore $\begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix} = f^2$. Similarly we can obtain other coefficients of x^2 and they will be a^2, b^2, c^2, d^2, e^2.

Thus the total coefficient of x^2 is $(a^2+b^2+c^2+d^2+e^2+f^2)$

The coefficient of x in the first row and first column is obtained by putting the three x's equal to zero in the determinant which is left by crossing the first row and find column and is therefore

$$\begin{vmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{vmatrix}$$ which is skew-symmetric determinant of odd order and is therefore zero **(Prop. 1, P. 112)**

Similarly all the coefficients of x^4 will be zero.

Clearly the coefficient of x^4 is unity.

$\therefore \ \Delta = x^4 + x^2 \ (a^2+b^2+c^2+d^2+e^2+f^2)+(af-be+cd)^2$.

Ex. 3. Expand the skew determinant

$$\begin{vmatrix} A & a & h & g & p \\ -a & B & b & f & n \\ -h & -b & C & c & r \\ -g & -f & -c & D & s \\ -p & -q & -r & -s & E \end{vmatrix}$$ **in terms of elements of the principal diagonal.**

(Agra M. Sc 19 50; Nagpur 61)

Clearly coefficient of $ABCDE$=unity.

There will be no term involving four of the five letters A, B, C, D, and E.

Coefficient of ABC is obtained by putting D and E equal to zero and is $\begin{vmatrix} 0 & s \\ -s & 0 \end{vmatrix} = s^2$. There will be ${}^5C_3$ such

coefficients, *i.e.* $\frac{5\,!}{2\,!\,3\,!}=10$ and they can be seen to be

$$a^2,\ b^2,\ c^2,\ f^2,\ g^2\ ;\ p^2,\ q^2,\ r^2 \text{ and } s^2.$$

Coefficient of AB is obtained be putting C, D and E all equal to zero and is

$$\begin{vmatrix} 0 & c & r \\ -c & 0 & s \\ -r & -s & 0 \end{vmatrix}$$

which being a skew-symmetric determinant of odd order is zero.

Therefore all other coefficients of the terms of the type AB etc. ${}^5C_2$ in number will be each zero.

Coefficient of A is obtained by putting B, C, D and E all equal to zero and is

$$\begin{vmatrix} 0 & b & f & q \\ -b & 0 & c & r \\ -f & -c & 0 & s \\ -q & -r & -s & 0 \end{vmatrix}$$

which being a skew-symmetric determinant of even order is a perfect square.

(Property 4 P. 116)

Its value is $\begin{matrix} b & f & q \\ c & r & s \end{matrix}$, *i.e.* $(bs-fr+cq)^2$ (See Rule P. 117).

There will be ${}^5C_1$ *i.e.* 5 such coefficients.

$$\therefore\ \Delta=ABCDE+\Sigma ABC.a^2+\Sigma A\,(bs-fr+cq)^2.$$

§ 28. Circulant Determinants.

Ex. 1. Explain the term circulant. Resolve the following determinant into linear factors.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{vmatrix}$$ (Nagpur M. Sc. 61)

Definition :—A determinant is said to be circulant if the constituent in each row be the same occuring in a circular order, but a different one standing first in each row.

In the above row, the last quantity of first row is a_5 and we start the 2nd row with a_5 followed by a_1, a_2, ..ending the row by a_4, and again start the 3rd row with a_4 followed by a_5, a_1, a_2, a_3 and so on.

Rule to solve. Here the determinant is of 5th order. Consider the equation $x^5-1=0$ and let its roots be 1, θ, θ^2, θ^3, θ^4 as we know that n nth roots of unity form a G. P. Also $\theta^5=1$.

(Dehli Hon's. 65)

Now take K for any of the above roots, *i.e.* K standing for 1, θ, θ^2, θ^3, θ^4. Add to the first column the constituents of remaining columns multiplied by K, K^2, K^3, K^4. Since K has five different values and you will find that each constituent of first column in each transformation is same. Thus you will have five factors.

Taking $K=1$, then K, K^2, K^3, K^4 are all equal to 1, hence we say adding all the columns to first column. It is easily seen that $a_1+a_2+a_3+a_4+a_5$ is a factor. ...(1)

Taking $K=\theta$, we say, apply $C_1+KC_2+K^2C_3+K^3C_4+K^4C_5$ *i.e.* $C_1+\theta C_2+\theta^2C_3+\theta^3C_4+\theta^4C_5$; the new first colun i becomes

$$\begin{array}{lc} a_1+\theta a_2+\theta^2 a_3+\theta^3 a_4+\theta^4 a_5 & 1 \\ a_5+\theta a_1+\theta^2 a_2+\theta^3 a_3+\theta^4 a_4 & \theta^4 \\ a_4+\theta a_5+\theta^2 a_1+\theta^3 a_2+\theta^4 a_3 & \theta^3 \\ a_3+\theta a_4+\theta^2 a_5+\theta^3 a_1+\theta^4 a_2 & \theta^2 \\ a_2+\theta a_3+\theta^2 a_4+\theta^3 a_5+\theta^4 a_1 & \theta \end{array}$$

If we devide and multiply each of the above by 1, θ^4, θ^3, θ^2, θ, respectively and put $\theta^5=1$, $\theta^6=\theta^5.\theta=\theta$, $\theta^8=\theta^5.\theta^3=\theta^3$ *i.e.* making the coefficient of a_1 unity we will observe that each element becomes

$$a_1+\theta a_2+\theta^2 a_3+\theta^3 a_4+\theta^4 a_5$$

showing thereby that it is a factor. ...(2)

Similarly taking $K=\theta^2$ we say

apply $C_1+KC_2+K^2C_3+K^3C_4+K^4C_5$

or $C_1+\theta^2C_2+\theta^4C_3+\theta C_4+\theta^3C_5$

$\because\ \theta^6=\theta^5.\theta=\theta$ and $\theta^8=\theta^5.\theta^3=\theta^3$

and proceeding as above we will find that each of the constituents of new first column is

$$a_1+\theta^2 a_2+\theta^4 a_3+\theta a_4+\theta^3 a_5 \qquad ...(3)$$

and hence it is a factor.

Similarly taking $K=\theta^3$ and θ^4 we will find the other factors.

$$\begin{aligned} \therefore\ \Delta=L\ &(a_1+a_2+a_3+a_4+a_5) \\ &(a_1+\theta a_2+\theta^2 a_3+\theta^3 a_4+\theta^4 a_5) \\ &(a_1+\theta^2 a_2+\theta^4 a_3+\theta a_4+\theta^3 a_5) \\ &(a_1+\theta^3 a_2+\theta a_3+\theta^4 a_4+\theta^2 a_5) \\ &(a_1+\theta^4 a_2+\theta^3 a_3+\theta^2 a_4+\theta a_5) \end{aligned}$$

Comparing the coefficient of a_1^5 in the given determinant (from diagonal terms) and in the product of above factors we find that $L=1$. Hence the factors.

Ex. 2. Resolve into factors the circulant

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Consider the equation $x^3-1=0$ and let its roots be 1, θ, θ^2 where $\theta^3=1$.

Let K stand for each of the above roots. Taking $K=1$, then apply $C_1+KC_2+K^2C_3$ *i.e.* $C_1+C_2+C_3$.

$$K^2=K=1, \text{ when } K=1.$$

The new first column gives $a+b+c$ as a factor. ...(1)

Taking $K=\theta$. we apply $C_1+KC_2+K^2C_3$ or $C_1+\theta C_2+\theta^2C_3$ the new first column becomes

$$\left.\begin{matrix} a+b\theta+c\theta^2 \\ c+a\theta+b\theta^2 \\ b+c\theta+a\theta^2 \end{matrix}\right\} \begin{matrix} 1 \\ \theta^2 \\ \theta. \end{matrix}$$

Multiplying and dividing by 1, θ^2, θ in order to make the coefficient of a unity, this column can be written as

$$a+b\theta+c\theta^2$$

$$\frac{1}{\theta^2}(a+b\theta+c\theta^2) \qquad\qquad \theta^4=\theta^3.\theta=\theta$$

$$\frac{1}{\theta}(a+b\theta+c\theta^2).$$

Thus we observe that $a+b\theta+c\theta^2$ is also a factor. ...(2)

Similarly taking $K=\theta^2$ and proceeding as above we find that $a+b\theta^2+c\theta$ is a factor.

$\therefore \quad \Delta=L\,(a+b+c)\,(a+b\theta+c\theta^2)\,(a+b\theta^2+c\theta).$

Comparing the coefficient of a^3 we find that $L=1$.

Ex. 3. If ω is a cube root of unity, prove that $a+b\omega+c\omega^2$ is a factor of

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

(Rajasthan 65)
(Agra 54)

$$\Delta=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=-\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

We have interchanged 2nd and 3rd rows and hence changed the sign and now the determinant is circulant and its factors as shown before are

$$(a+b+c)\,(a+b\theta+c\theta^2)\,(a+b\theta^2+c\theta)$$

$$\therefore\quad \Delta=-(a+b+c)\,(a+b\theta+c\theta^2)\,(a+b\theta^2+c\theta)$$

where 1, θ, θ^2 are the roots of $x^3-1=0$ which are given to be 1, ω, ω^2,

Ex 4. Prove that the product of two circulants of the same order is a circulant. **(Nagpur M. Sc. 61)**

Consider the product of two circulants

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}\times\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

$$=\begin{vmatrix} ax+by+cz & az+bx+cy & ay+bz+cx \\ cx+ay+bz & cz+ax+by & cy+az+bx \\ bx+cy+az & bz+cx+ay & by+cz+ax \end{vmatrix}$$

If we put $\quad ax+by+cz=A,$

$$bx+cy+az=B,$$

$$cx+ay+bz=C,$$

then the above determinant is

$$\begin{vmatrix} A & B & C \\ C & A & B \\ B & C & A \end{vmatrix}$$

which is a circulant.

Exercise 2

Q. 1. Prove that

$$\begin{vmatrix} 1 & x & y & z \\ -x & 1 & z' & -y' \\ -y & -z' & 1 & x' \\ -z & y' & -x' & 1 \end{vmatrix} = 1+x^2+y^2+z^2+x'^2+y'^2+z'^2 + (xx'+yy'+zz')^2$$

Expand in terms of minors formed from the first two columns. [§ 22 P, 97]

$$\begin{vmatrix} 1 & x \\ -x & 1 \end{vmatrix} \begin{vmatrix} 1 & x' \\ -x' & 1 \end{vmatrix} - \begin{vmatrix} 1 & x \\ -y & -z' \end{vmatrix} \begin{vmatrix} z' & -y' \\ -x' & 1 \end{vmatrix}$$
$$+ \begin{vmatrix} 1 & x \\ -z & y' \end{vmatrix} \begin{vmatrix} z' & -y' \\ 1 & x' \end{vmatrix} + \begin{vmatrix} -x & -z' \\ -y & -z' \end{vmatrix} \begin{vmatrix} y & z \\ -x' & 1 \end{vmatrix}$$
$$- \begin{vmatrix} -x & 1 \\ -z & y' \end{vmatrix} \begin{vmatrix} y & z \\ 1 & x' \end{vmatrix} + \begin{vmatrix} -y & -z' \\ -z & y' \end{vmatrix} \begin{vmatrix} y & z \\ z' & -y' \end{vmatrix}$$

Now simplify etc.

Q. 1. Prove that

$$\begin{vmatrix} -\alpha & \beta & \gamma & \delta \\ \beta & -\alpha & \delta & \gamma \\ \gamma & \delta & -\alpha & \beta \\ \delta & \gamma & \beta & -\alpha \end{vmatrix}.$$

$$=\alpha^4+\beta^4+\gamma^4+\delta^4-2\alpha^2\beta^2-2\alpha^2\gamma^2-2\alpha^2\delta^2-2\beta^2\gamma^2-2\beta^2\delta^2-2\gamma^2\delta^2-8\alpha\beta\gamma\delta.$$

Expand as above in Q. 1, in terms of minors from the first two columns

Q. 3. Prove the following identity and expand :=

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$$

$$=x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2.$$

Multiply C_1 by xyz, C_2 by x, C_3 by y and C_4 by z and therefore

divide by $\dfrac{1}{xyz.\,x\,y.\,z}$

$$\Delta=\frac{1}{x^2y^2z^2}\begin{vmatrix} 0 & x & y & z \\ xyz & 0 & yz^2 & y^2z \\ xyz & xz^2 & 0 & zx^2 \\ xyz & xy^2 & yx^2 & 0 \end{vmatrix}$$

Now take yz from R_2, zx from R_3 and xy from R_4.

$$\Delta=\frac{xy\,yz.zx}{x^2y^2z^2}\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}=\text{R. H. S.}$$

In order to find the value, proced as in Q. 1 and 2.

Note. The two determinants have been separately evaluated before in **Q. 44.** and **45 P. 48** and **49** respectively.

Q. 4. Prove that

$$\begin{vmatrix} a & h & g & x & x' \\ h & b & f & y & y' \\ g & f & c & z & z' \\ x & y & z & 0 & 0 \\ x' & y' & z' & 0 & 0 \end{vmatrix}$$

$$=a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta,$$

where $\alpha=yz'-y'z$, $\beta=zx'-z'x$ and $\gamma=xy'-x'y$.

(Nagpur 61)

Here C_4 and C_5 can be made C_1 and C_2 by $(3+3)$, *i.e.* 6 movements of columns thereby causing no change and as such we expand the determinant as above in terms of minors of C_4 and C_5, and we get

$$\begin{vmatrix} x & x' \\ y & y' \end{vmatrix}\begin{vmatrix} g & f & c \\ x & y & z \\ x' & y' & z' \end{vmatrix}-\begin{vmatrix} x & x' \\ z & z' \end{vmatrix}\begin{vmatrix} h & b & f \\ x & y & z \\ x' & y' & z' \end{vmatrix}$$

$$+\begin{vmatrix} y & y' \\ z & z' \end{vmatrix}\begin{vmatrix} a & h & g \\ x & y & z \\ x' & y' & z' \end{vmatrix}$$

All other determinants vanish as each determinant of 2nd order will have a row of zeros. Now expand and put the values of α, β and γ.

$$\gamma\,[g\alpha-f(-\beta)+c\gamma]-(-\beta)\,\{h\alpha-b\,(-\beta)+f\gamma]+\alpha[a\alpha-h(-\beta)+g\gamma$$

$$=a\alpha^2+b\beta^2+c\gamma^2+2\,f\beta\gamma+2g\gamma\alpha+2h\alpha\beta.$$

Q. 5. If $f_1(x)=a_1x^3+3b_1x^2+3c_1x+d_1$,
$f_2(x)=a_2x^3+3b_2x^2+3c_2x+d_2$,
$f_3(x)=a_3x^3+3b_3x^2+3c_3x+d_3$,

prove the identity

$$\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix} = -18 \begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}.$$

(Agra M.Sc. 53)

$$\Delta = \begin{vmatrix} a_1x^3+3b_1x^2+3c_1x+d_1 & 3\,(a_1x^2+2b_1x+c_1) & 6\,(a_1x+b_1) \\ a_2x^3+3b_2x^2+3c_2x+d_2 & 3\,(a_2x^2+2b_2x+c_2) & 6\,(a_2x+b_2) \\ a_3x^3+3b_3x^2+3c_3x+d_3 & 3\,(a_3x^2+2b_3x+c_3) & 6\,(a_3x+b_3) \end{vmatrix}$$

Take 3 and 6 common from C_2 and C_3 and then apply C_2-xC_3.

$$\Delta = 18 \begin{vmatrix} a_1x^3+3b_1x^2+3c_1x+d_1 & b_1x+c_1 & a_1x+b_1 \\ a_2x^3+3b_2x^2+3c_2x+d_2 & b_2x+c_2 & a_2x+b_2 \\ a_3x^3+3b_3x^2+3c_3x+d_3 & b_3x+c_3 & a_3x+b_3 \end{vmatrix}$$

Now apply $C_1-x^2C_3-2xC_2$ and then interchange C_1 and C_3.

$$\Delta = -18 \begin{vmatrix} a_1x+b_1 & b_1x+c_1 & c_1x+d_1 \\ a_2x+b_2 & b_2x+c_2 & c_2x+d_2 \\ a_3x+b_3 & b_3x+c_3 & c_3x+d_3 \end{vmatrix}$$

The deteminant on R.H.S. is of fourth order and as such Δ can be written as

$$\Delta=-18\begin{vmatrix}1 & 0 & 0 & 0\\ a_1 & a_1x+b_1 & b_1x+c_1 & c_1x+d_1\\ a_2 & a_2x+b_2 & b_2x+c_2 & c_2x+d_2\\ a_3 & a_3x+b_3 & b_3x+c_3 & c_3x+d_3\end{vmatrix}$$

Now apply C_2-xC_1 and then in the new determinant thus obtained again C_3-xC_2 and then again in the new determinant thus obtained C_4-xC_3 and you will get the determinant on R.H.S.

Q. 6. (a) If $f(x)=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$, then prove that

$$\begin{vmatrix}\alpha & x & x & x\\ x & \beta & x & x\\ x & x & \gamma & x\\ x & x & x & \delta\end{vmatrix}=f(x)-xf'(x).$$

(Agra M. Sc. 44, 51, 55 ; Delhi Hons. 56

Taking log of both sides of the given relation, we get

$\log f(x)=\log(x-\alpha)+\log(x-\beta)+\log(x-\gamma)+\log(x-\delta)$.

Differentiating we get $f'(x)=\dfrac{f(x)}{x-\alpha}+\dfrac{f(x)}{x-\beta}+\dfrac{f(x)}{x-\gamma}+\dfrac{f(x)}{x-\delta}$

Now applying R_1-R_2, R_2-R_3, R_3-R_4 on Δ, we get

$$\Delta=\begin{vmatrix}\alpha-x & x-\beta & 0 & 0\\ 0 & \beta-x & x-\gamma & 0\\ 0 & 0 & \gamma-x & x-\delta\\ x & x & x & \delta\end{vmatrix}$$

$$=(\alpha-x)\begin{vmatrix}\beta-x & x-\gamma & 0\\ 0 & \gamma-x & x-\delta\\ x & x & \delta\end{vmatrix}-x\begin{vmatrix}x-\beta & 0 & 0\\ \beta-x & x-\gamma & 0\\ 0 & \gamma-x & x-\delta\end{vmatrix}$$

$$=(\alpha-x)\left[(\beta-x)\{(\gamma-x)\delta-x(x-\delta)\}+x(x-\gamma)(x-\delta)\right]$$
$$-x(x-\beta)(x-\gamma)(x-\delta).$$

Collecting the terms of x and writing each factor as $x-\alpha$ etc., i.e. $\alpha-x$ to be written as $-(x-\alpha)$

$$=(x-\alpha)(x-\beta)(x-\gamma)[x-\delta-x]-x[(x-\alpha)(x-\beta)(x-\delta)+(x-\alpha)(x-\gamma)(x-\delta)+(x-\beta)(x-\gamma)(x-\delta)].$$

Taking $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ common,

$$=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\left[1-x\left(\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma}+\frac{1}{x-\delta}\right)\right]$$

$$=f(x)-x\left(\frac{f(x)}{x-\alpha}+\frac{f(x)}{x-\beta}+\frac{f(x)}{x-\gamma}+\frac{f(x)}{x-\delta}\right)$$

$=f(x)-xf'(x)$ by (1).

(b) If $f(x)=(a_1-x)(a_2-x)(a_3-x)(a_4-x)(a_5-x)$, show that

$$\begin{vmatrix} a_1 & x & x & x & x \\ x & a_2 & x & x & x \\ x & x & a_3 & x & x \\ x & x & x & a_4 & x \\ x & x & x & x & a_5 \end{vmatrix}=f(x)-x\frac{df}{dx}$$

(Agra M. Sc. 61)

Apply $R_1-R_2,\ R_2-R_3,\ R_3-R_4,\ R_4-R_5$.

$$\Delta=\begin{vmatrix} a_1-x & x-a_2 & 0 & 0 & 0 \\ 0 & a_2-x & x-a_3 & 0 & 0 \\ 0 & 0 & a_3-x & x-a_4 & 0 \\ 0 & 0 & 0 & a_4-x & x-a_5 \\ x & x & x & x & a_5 \end{vmatrix}$$

Expanding with first column,

$$
\Delta=(a_1-x)\begin{vmatrix} a_2-x & x-a_3 & 0 & 0 \\ 0 & a_3-x & x-a_4 & 0 \\ 0 & 0 & a_4-x & x-a_5 \\ x & x & x & a_5 \end{vmatrix}
+x\begin{vmatrix} x-a_2 & 0 & 0 & 0 \\ a_2-x & x-a_3 & 0 & 0 \\ 0 & a_3-x & x-a_4 & 0 \\ 0 & 0 & a_4-x & x-a_5 \end{vmatrix}
$$

$$
\begin{aligned}
&=(a_1-x)\left\{(a_2-x)\begin{vmatrix} a_3-x & x-a_4 & 0 \\ 0 & a_4-x & x-a_5 \\ x & x & a_5 \end{vmatrix}
-x\begin{vmatrix} x-a_3 & 0 & 0 \\ a_3-x & x-a_4 & 0 \\ 0 & a_4-x & x-a_5 \end{vmatrix}\right\} \\
&\qquad +x\,(x-a_2)\,(x-a_3)\,(x-a_4)\,(x-a_5) \\
&=(a_1-x)\,(a_2-x)\,(a_3-x)\,\{a_5\,(a_4-x)-x\,(x-a_5)\} \\
&\qquad +x\,(x-a_4)\,(x-a_5)] \\
&\qquad -x\,(a_1-x)\,[(x-a_3)\,(x-a_4)\,(x-a_5)] \\
&\qquad +x\,(x-a_2)\,(x-a_3)\,(x-a_4)\,(x-a_5).
\end{aligned}
$$

Collecting the terms of x and writing each factor as $a-x$ instead of $x-a$ and thus adjusting the sign,

$$
\begin{aligned}
\Delta=&(a_1-x)\,(a_2-x)\,(a_3-x)\,(a_4-x)\,[(a_5-x)+x] \\
&+x\,[(a_1-x)\,(a_2-x)\,(a_3-x)\,(a_5-x) \\
&+(a_1-x)\,(a_2-x)\,(a_4-x)\,(a_5-x) \\
&+(a_1-x)\,(a_3-x)\,(a_4-x)\,(a_5-x) \\
&+(a_2-x)\,(a_3-x)\,(a_4-x)\,(a_5-x)].
\end{aligned}
$$

Now take $(a_1-x)(a_2-x)(a_3-x)(a_4-x)(a_5-x)$ common.

$$\therefore \quad \Delta=f(x)\left[1+x\left(\frac{1}{a_1-x}+\frac{1}{a_2-x}+\frac{1}{a_3-x}+\frac{1}{a_4-x}+\frac{1}{a_5-x}\right)\right]$$
$$=f(x)-x\,f'(x)$$

$\because$ As in part (a) taking log,

$$f'(x)=-\left[\frac{f(x)}{a_1-x}+\frac{f(x)}{a_2-x}+\frac{f(x)}{a_3-x}+\frac{f(x)}{a_4-x}+\frac{f(x)}{a_5-x}\right]$$

Q. 7. Show that the determinant

$$\begin{vmatrix} \lambda x^2+cy^2+bz^2-1 & (\lambda-c)\,xy & (\lambda-b)\,xz \\ (\lambda-c)\,xy & \lambda y^2+az^2+cx^2-1 & (\lambda-a)\,yz \\ (\lambda-b)\,xz & (\lambda-a)\,yz & \lambda z^2+bx^2+ay^2-1 \end{vmatrix}$$

contains $\lambda(x^2+y^2+z^2)-1$ as a fector and show also that the remaining factor is independent of λ

The given determinant can be made of fourth order without changing its value.

i.e.
$$\Delta=\begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda x & \begin{matrix}\lambda x^2+cy^2\\+bz^2-1\end{matrix} & (\lambda-c)\,xy & (\lambda-b)\,xz \\ \lambda y & (\lambda-c)xy & \begin{matrix}\lambda y^2+az^2\\+cx^2-1\end{matrix} & (\lambda-a)\,yz \\ \lambda z & (\lambda-b)\,xz & (\lambda-a)\,yz & \begin{matrix}\lambda z^2+bx^2\\+ay^2-1\end{matrix} \end{vmatrix}$$

Apply $C_2-xC_1,\ C_3-yC_1,\ C_4-zC_1$.

$$\therefore\ \Delta=\begin{vmatrix} 1 & -x & -y & -z \\ \lambda x & cy^2+bz^2-1 & -cxy & -bxz \\ \lambda y & -cxy & az^2+cx^2-1 & -ayz \\ \lambda z & -bxz & -ayz & bx^2+ay^2-1 \end{vmatrix}$$

Now apply $R_1-xR_2-yR_3-zR_4$, thus making the first row of the new determinant as

$$1-\lambda\ (x^2+y^2+z^2)\quad 0\quad 0\quad 0$$

all other constituents remaining unchanged Now expanding with first row,

$$\Delta=[1-\lambda\ (x^2+y^2+z^2)]\times\begin{vmatrix} cy^2+bz^2-1 & -cxy & -bxz \\ -cxy & az^2+cx^2-1 & -ayz \\ -bxz & -ayz & bx^2+ay^2-1 \end{vmatrix}.$$

Above form shows that Δ has a factor $\lambda\ (x^2+y^2+z^2)-1$ and that the remaining factor is independent of λ.

Q. 8. Show that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}=(a^2+b^2+c^2)\,(a+b+c)\times(a-b)\,(b-c)\,(c-a).$$

(Agra 34)

Proceed as in Q. 18 P. 85.

***Q. 9.* Utilize Laplace's Method of expanding a determinant to prove that the product of two determinants can be put as a single determinant.**

(Agra M.Sc. 42, 53)

Consider the determinant

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & -1 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & -1 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \quad \ldots(1)$$

$$\times\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}. \qquad \ldots (B)$$

The latter determinant is clearly -1. Hence we have from (A) and (B) by changing the rows into columns and columns into rows.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1\alpha_1+b_1\alpha_2+c_1\alpha_3 & a_1\beta_1+b_1\beta_2+c_1\beta_3 & a_1\gamma_1+b_1\gamma_2+c_1\gamma_3 \\ a_2\alpha_1+b_2\alpha_2+c_2\alpha_3 & a_2\beta_1+b_2\beta_2+c_2\beta_3 & a_2\gamma_1+b_2\gamma_2+c_2\gamma_3 \\ a_3\alpha_1+b_3\alpha_2+c_3\alpha_3 & a_3\beta_1+b_3\beta_2+c_3\beta_3 & a_3\gamma_1+b_3\gamma_2+c_3\gamma_3 \end{vmatrix}.$$

Q. 10. Evaluate

$$\begin{vmatrix} x & x^2 & x^3 & \ldots & \ldots & x^{n-2} & x^{n-1} & x^n \\ x^2 & x^3 & x^4 & \ldots & \ldots & x^{n-1} & x^n & x \\ x^3 & x^4 & x^5 & \ldots & \ldots & x^n & x & x^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^n & x & x^2 & \ldots & \ldots & x^{n-3} & x^{n-2} & x^{n-1} \end{vmatrix}$$

Apply C_n-xC_{n-1}, $C_{n-1}-xC_{n-2}$, $C_{n-2}-xC_{n-3}$ $\ldots C_3-xC_2$, C_2-xC_1.

$$\Delta=\begin{vmatrix} x & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 \\ x^2 & 0 & 0 & \ldots & \ldots & 0 & 0 & x(1-x^n) \\ x^3 & 0 & 0 & \ldots & \ldots & 0 & x(1-x^n) & 0 \\ x^4 & 0 & 0 & \ldots & \ldots & x(1-x^n) & 0 & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ x^{n-1} & 0 & x(1-x^n) & \ldots & \ldots & 0 & 0 & 0 \\ x^n & x(1-x^n) & 0 & \ldots & & 0 & 0 & 0 \end{vmatrix}$$

$$=x\begin{vmatrix} 0 & 0 & \ldots & 0 & 0 & x(1-x^n) \\ 0 & 0 & \ldots & 0 & x(1-x^n) & 0 \\ 0 & 0 & \ldots & x(1-x^n) & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & x(1-x^n) & . & 0 & 0 & 0 \\ x(1-x^n) & 0 & . & 0 & 0 & 0 \end{vmatrix}$$

which is a determinant of $(n-1)$th order.

Now expand it with the last column *i.e.* $(n-1)$th which can be made first column by $(n-2)$ movements and hence we will have to attach $(-1)^{n-2}$. Again we will be left with a determinant of $(n-2)$th order which has got to be expanded again with its last *i.e.* $(n-2)$th column. This last can be made first by $(n-3)$ movements and hence we will have to attach $(-1)^{n-3}$ and so on.

$$\therefore\quad \Delta=x\,(-1)^{n-2}\,[x\,(1-x^n)]\,(-1)^{n-3}\,[x\,(1-x^n)]$$
$$\times(-1)^{n-4}\,[x(1-x^n)]\ldots(n-1)\text{ times}$$
$$=(-1)^{(n-2)+(n-3)+(n-4)+\ldots 3+2+1}x.\,[x(1-x^n)]^{n-1}$$
$$=(-1)^{(n-2)\,(n-1)/2}x^n.\,(1-x^n)^{n-1},\ \because\ \Sigma n=n\,(n+1)/2.$$

Q. 11. (a) Evaluate

$$\begin{vmatrix} a_n & b_n & 0 & 0 & 0 & \ldots & 0 & 0 \\ -1 & a_{n-1} & b_{n-1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & a_{n-2} & b_{n-2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & -1 & a_{n-3} & b_{n-3} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & a_2 & b_2 \\ 0 & 0 & 0 & 0 & 0 & \ldots & -1 & a_1 \end{vmatrix}$$

The above determinant is called continuant determinant. All the diagonal elements are a's, the line above the diagonal is of b's and that below the diagonal is of -1. Let us denote the determinant of nth order by Δ_n. Expanding with first row (not with first column), we get

$$\Delta_n = a_n\Delta_{n-1} - b_n\cdot\{-1\,(\Delta_{n-2}) + 0\} + 0$$

or

$$\Delta_n = a_n\Delta_{n-1} + b_n\Delta_{n-2}.$$

$$\therefore \quad \frac{\Delta_n}{\Delta_{n-1}} = a_n + b_n\,\frac{\Delta_{n-2}}{\Delta_{n-1}}$$

or

$$\frac{\Delta_n}{\Delta_{n-1}} = a_n + \cfrac{b_n}{\cfrac{\Delta_{n-1}}{\Delta_{n-2}}} \qquad \ldots(A)$$

$$= a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{\cfrac{\Delta_{n-2}}{\Delta_{n-3}}}} \qquad [\text{by } (A)]$$

$$= a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{a_{n-2} + \cfrac{b_{n-2}}{\cfrac{\Delta_{n-3}}{\Delta_{n-4}}}}} \quad [\text{by } (4)].$$

Now $\Delta_2 = \begin{vmatrix} a_2 & b_2 \\ -1 & a_1 \end{vmatrix} = a_1a_2 + b_2$ and $\Delta_1 = a_1$.

$$\therefore \quad \frac{\Delta_2}{\Delta_1} = \frac{a_1a_2 + b_2}{a_1} = a_2 + \frac{b_2}{a_1}.$$

Hence continuing the above process till at last we reach to the calculation of $\frac{\Delta_2}{\Delta_1}$ which is $a_2 + \frac{b_2}{a_1}$. Hence in terms of continued fraction we can write

$$\frac{\Delta_n}{\Delta_{n-1}} = a_n + \frac{b_n}{c_{n-1}+}\,\frac{b_{n-1}}{c_{n-2}+}\cdots\frac{b_3}{+a_2+}\,\frac{b_2}{a_1}.$$

Note. If all the b's be replaced by $+1$, then the above determinant will be called simple continuant determinant and in that case

$$\frac{\Delta_n}{\Delta_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1}.$$

(Agra M.Sc. 48)

(b) Prove that the value of the determinant

$$\begin{vmatrix} 1+x^2 & x & 0 & 0 & \dots & 0 \\ x & 1+x^2 & x & 0 & \dots & 0 \\ 0 & x & 1+x^2 & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & x & 1+x^2 \end{vmatrix}$$

of mth order is $1+x^2+x^4+\dots+x^{2m}$. (I. A. S. 59)

$$\Delta_1=1+x^2,\ \Delta_2=\begin{vmatrix} 1+x^2 & x \\ x & 1+x^2 \end{vmatrix}=(1+x^2)^2-x^2 =1+x^2+x^4.$$

$$\Delta_3=\begin{vmatrix} 1+x^2 & x & 0 \\ x & 1+x^2 & x \\ 0 & x & 1+x^2 \end{vmatrix}=(1+x^2)\,\Delta_2-x\begin{vmatrix} x & 0 \\ x & 1+x^2 \end{vmatrix}$$

$$=(1+x^2)(1+x^2+x^4)-x.x\,(1+x^2)$$

$$=(1+x^2)\,[1+x^2+x^4-x^2]$$

$$=(1+x^2)\,(1+x^4)=1+x^2+x^4+x^6.$$

Again $\Delta_m=(1+x^2)\,\Delta_{m-1}-$

$$x\begin{vmatrix} x & x & 0 & \dots & 0 \\ 0 & 1+x^2 & x & \dots & 0 \\ 0 & x & 1+x^2 & \dots & 0 \\ 0 & \dots & \dots & \dots & 1+x^2 \end{vmatrix}$$

$(m-1)$th order

or $$\Delta_m=(1+x^2)\,\Delta_{m-1}-x.x.\Delta_{m-2}$$

or $$\Delta_m=(1+x^2)\,\Delta_{m-1}-x^2.\Delta_{m-2}.$$

Putting $m=4$, we get

$\Delta_4=(1+x^2)\,\Delta_3-x^2.\Delta_2.$

$=(1+x^2)\,(1+x^2+x^4+x^6)-x^2\,(1+x^2+x^4)$

$=(1+x^2)\,x^6+(1+x^2+x^4)\,(1+x^2-x^2)=1+x^2+x^4+x^6+x^8.$

Similarly we can show that

$$\Delta_5=1+x^2+x^4+x^6+x^8+x^{10}$$

and $$\Delta_m=1+x^2+x^4+x^6+\ldots+x^{2m}.$$ Proved.

Q. 11. (e) Prove that the determinant of nth order.

$$\begin{vmatrix} \cos\theta & 1 & 0 & 0 & \ldots & \ldots & 0 \\ 1 & 2\cos\theta & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 2\cos\theta & 1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 1 & 2\cos\theta & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\cos\theta \end{vmatrix}=\cos n\theta.$$

$\Delta_1=\cos\theta,\ \Delta_2=\begin{vmatrix} \cos\theta & 1 \\ 1 & 2\cos\theta \end{vmatrix}$

or $\Delta_2=2\cos^2\theta-1=\cos 2\theta$

$$\Delta_3=\begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix}$$

$=\cos\theta\,(4\cos^2\theta-1)-1\,(2\cos\theta)$

$4\cos^3\theta-3\cos\theta=\cos 3\theta.$

$$\Delta_4=\begin{vmatrix} \cos\theta & 1 & 0 & 0 \\ 1 & 2\cos\theta & 1 & 0 \\ 0 & 1 & 2\cos\theta & 1 \\ 0 & 0 & 1 & 2\cos\theta \end{vmatrix}.$$

Expanding with last column,

$$-\left\{1\begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 0 & 1 \end{vmatrix} -2\cos\theta\begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix}\right\}$$

$$-1\,[1\,(2\cos^2\theta-1)]+2\cos\theta.\Delta_3.$$

Now put $\Delta_3=\cos 3\theta$.

$$\Delta_4=-(2\cos^2\theta-1)+2\cos\theta\cos 3\theta$$

$$=-\cos 2\theta+\cos 4\theta+\cos 2\theta=\cos 4\theta.$$

Similarly we can prove that $\Delta_5=\cos 5\theta$.

Hence $\Delta_n=\cos n\theta$.

Q. 12. Evaluate

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \dots & x^{n-1} \\ x & 1 & 0 & 0 & \dots & 0 \\ x^2 & 0 & 1 & 0 & \dots & 0 \\ x^3 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x^{n-1} & 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

expanding with first column or first row.

(Vikram M.Sc. 59 ; Agra M.Sc. 1936, 60)

$\Delta=1.$(a determinant whose value is clearly unity)

$-x$ (a determinant whose value is clearly x)

$$+x^2\begin{vmatrix} x & x^2 & x^3 & \dots & \dots \\ 1 & 0 & 0 & \dots & .. \\ 0 & 0 & 1 & \dots & .. \\ 0 & 0 & 0 & \dots & .. \\ \dots & \dots & \dots & \dots & .. \end{vmatrix}$$

i.e. $+x^2\left\{x(0)-1\begin{vmatrix} x^2 & x^3 & \dots \\ 0 & 1 & \dots \\ 0 & 0 & \dots \end{vmatrix}\right\}$ *i.e.* $x^2(-x^2)=-x^4$

$$-x^3\begin{vmatrix} x & x^2 & x^3 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \end{vmatrix}$$

i.e. $-x^3\left\{x(0)-1\begin{vmatrix} x^2 & x^3 \\ 1 & 0 \end{vmatrix}\right\}$ *i.e.* $-x^3(x^3)=-x^6$

and so on,

$$\therefore\quad \Delta=1-x^2-x^4-x^6 \dots x^{2n-2}$$

$$=1-\{x^2+x^4+x^6+\dots(n-1)\text{ terms}\}$$

$$=1-\frac{x^2[(x^2)^{n-1}-1]}{x^2-1}\quad\because\quad \frac{a(r^n-1)}{r-1}=\text{sum of a G. P.}$$

or $$\frac{x^2-1-x^{2n}+x^2}{x^2-1}=\frac{2x^2-x^{2n}-1}{x^2-1}.$$

Alternative method.

In case there be no restriction to expand the determinant either with first column or first row we can evaluate the above determinant as below.

Multiply $C_2, C_3 \; C_n$ by $-x, -x^2 \ldots -x^{n-1}$ respectively,

$$\therefore \quad \Delta = \frac{1}{(-x)\,(-x^2)\ldots(-x^{n-1})}$$

$$\times \begin{vmatrix} 1 & -x^2 & -x^4 & \ldots & \ldots & -x^{2n-2} \\ x & -x & 0 & \ldots & \ldots & 0 \\ x^2 & 0 & -x^2 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^{n-1} & 0 & 0 & 0 & 0 & -x^{n-1} \end{vmatrix}$$

Add all the columns to fi st column and you will find that new first column will have zero everywhere except the first constituent which will be

$$1 - x^2 - x^4 \ldots - x^{2n-2}.$$

Expanding with this first column,

$$\Delta = \frac{1 - x^2 - x^4 \ldots - x^{2n-2}}{(-x)\,(-x^2)\ldots(-x^{n-1})}$$

$$\begin{vmatrix} -x & 0 & \ldots & \ldots & 0 \\ 0 & -x^2 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & -x^{n-1} \end{vmatrix}$$

The above determinant of $(n-1)$th order and is equal to product of the terms in principal diagonal.

$$\therefore \quad \Delta = \frac{1 - x^2 - x^4 \ldots - x^{2n-2}}{(-x)\,(-x^2)\ldots(-x^{n-1})} \times (-x)\;(-x^2)\ldots(-x^{n-1})$$

$$= 1 - x^2 - x^4 \ldots - x^{2n-2}.$$

$$=1-x^2\frac{\{(\lambda^2)^{n-1}-1\}}{(x^2-1)} \text{ summing a G.P.}$$

$$=\frac{x^2-1-x^{2n}+x^2}{x^2-1}=\frac{2\lambda^2-x^{2n}-1}{x^2-1} \qquad \text{Proved.}$$

Q. 13. Find the condition that a biquadratic should be capable of being put as sum of two fourth powers and expressing it in the form

$$ax^4+4bx^3+6cx^2+4dx+e=A\ (x+\theta)^4+B\ (x+\phi)^4.$$

find the quadratic whose roots are θ and ϕ.

(Agra M. Sc. 42)

On comparing the coefficients, we have

$$A+B=a, \qquad \ldots(1)$$

$$A\theta+\beta\phi=b, \qquad \ldots(2)$$

$$A\theta^2+B\phi^2=c, \qquad \ldots(3)$$

$$B\theta^3+B\phi^3=d, \qquad \ldots(4)$$

$$A\theta^4+B\phi^4=e, \qquad \ldots(5)$$

Now assume that θ and ϕ are the roots of

$$x^2+\lambda x+\mu=0 \qquad \ldots(6)$$

so that $$\theta^2+\lambda\theta+\mu=0 \qquad \ldots(7)$$

and $$\phi^2+\lambda\phi+\mu=0. \qquad \ldots(8)$$

Now multiply (3) by 1, (2) by λ and (1) by μ and add

$$\therefore \quad A\ (\theta^2+\lambda\theta+\mu)+B\ (\phi^2+\lambda\phi+\mu)=c+b\lambda+a\mu.$$

$$\therefore \quad c+b\lambda+a\mu=0 \qquad \ldots(9)$$

from (7) and (8)

Similarly $$d+c\lambda+b\mu=0 \qquad \ldots(10)$$

and $$e+d\lambda+c\mu=0.$$

Eliminating λ and μ between (9), (10) and (11), we have the required condition as

$$\begin{vmatrix} c & b & a \\ d & c & d \\ e & b & c \end{vmatrix}=0 \text{ or } \begin{vmatrix} a & b & c \\ b & e & d \\ c & d & e \end{vmatrix}=0 \quad \text{or} \quad J=0.$$

Also the quadratic equation whose roots are θ and ϕ is obtained by eliminating λ and μ between (6) and (9) and (10) and is

$$\begin{vmatrix} \lambda^2 & x & 1 \\ c & b & a \\ d & c & b \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1 & x & x^2 \\ a & b & c \\ b & c & d \end{vmatrix} = 0$$

Note: – If the cubic were expressed as sum of two cubes as above, then, also proceeding as above we would obtain the same quadratic giving the values of θ, ϕ.

[See Author's book on Theory of Equations P. 121]

Q. 14. If
$$\begin{vmatrix} a & b & c & d & 0 & 0 \\ a' & b' & c' & d' & 0 & 0 \\ a'' & b'' & c'' & d'' & 0 & 0 \\ 0 & a & b & 0 & c & d \\ 0 & a' & b' & 0 & c' & d' \\ 0 & a'' & b'' & 0 & c'' & d'' \end{vmatrix}$$

prove that
$$\begin{vmatrix} a & c & d \\ a' & c' & d' \\ a'' & c'' & d'' \end{vmatrix}^2 = \begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix} \begin{vmatrix} b & c & d \\ b' & c' & d' \\ b'' & c'' & d'' \end{vmatrix}$$

(Agra M. Sc. 39)

Expand the given determinant in terms of minors formed from 1st three rows and you will get so many determinants the first of which will be

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \begin{vmatrix} 0 & c & d \\ 0 & c' & d' \\ 0 & c'' & '' \end{vmatrix}$$

which is zero. Similarly you will find that in all the

factors three will be a determinant with one line of zeros, and hence zero except in the terms where given determinants occur, etc.

Q. 15.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

If capital letters stand for the cofactors of the corresponding small letters in the above determinant, then prove that

$$\Delta=\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} \qquad \textbf{(Rajputana 60)}$$

$$=-[Al^2+Bm^2+cn^2+2Fmm+2Gnl+2Hlm]$$

(Refer Anthor's Solid Geometry P. 239)

Expanding with respect to last column or last row, we get

$$\Delta=(-1)\left\{l\begin{vmatrix} h & b & f \\ g & f & c \\ l & m & n \end{vmatrix}-m\begin{vmatrix} a & h & g \\ g & f & c \\ l & m & n \end{vmatrix}+n\begin{vmatrix} a & h & g \\ h & b & f \\ l & m & n \end{vmatrix}+0\right\}$$

Now $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=aA+hH+gG=hH+bB+fF.$
$=gG+fF+cC.$

$\therefore \begin{vmatrix} h & b & f \\ g & f & c \\ l & m & n \end{vmatrix}=\begin{vmatrix} l & m & n \\ h & b & f \\ g & f & c \end{vmatrix}=lA+mH+nC$ *i.e.* we have replaced a, h, g by l, m, n, the other rows giving the cofactors remain unchanged and hence capital letters remain unchanged.

$$\begin{vmatrix} a & h & g \\ g & f & c \\ l & m & n \end{vmatrix} = -\begin{vmatrix} a & h & g \\ l & m & n \\ g & f & c \end{vmatrix} = -(lH+mB+nF)$$
as above.

$$\begin{vmatrix} a & h & g \\ h & f & b \\ l & m & n \end{vmatrix} = lG+mF+nC.$$

$$\therefore \quad \Delta = (-1)\,[l(lA+mH+nG)+m\,(lH+mB+nF) + n\,(lG+mF+nC)]$$
$$= -(Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm).$$

Hence proved.

AGRA UNIVERSITY M. A. & M. Sc. PAPERS

1960

(a) Define skew-symmetric and skew determinants and establish the following properties :

(i) A skew-symmetric determinant of odd order vanishes.

(ii) The reciprocal of a skew-symmetric determinant of even order is a skew-symmetric determinant.

(b) Evaluate

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \dots & x^{n-1} \\ x & 1 & 0 & 0 & \dots & 0 \\ x^2 & 0 & 1 & 0 & \dots & 0 \\ x^3 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x^{n-1} & 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

1961

1. (a) If $f(x)=(a_1-x)(a_2-x)(a_3-x)(a_4-x)(a_5-x)$, show that

$$\begin{vmatrix} a_1 & x & x & x & x \\ x & a_2 & x & x & x \\ x & x & a_3 & x & x \\ x & x & x & a_4 & x \\ x & x & x & x & a_5 \end{vmatrix} = f(x) - x\frac{df}{dx}.$$

(b) Prove that if in any symmetrical determinant, the leading first minor vanishes, the determinant itself and its leading second minor have opposite signs.

1962

(c) Define skew-symmetric and skew determinants. Prove that every skew-symmetric determinant of even order is a perfect square.

(b) Prove the identity.

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

1963

(c) Prove that $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha+\beta+\gamma)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta)$

(d) Define a symmetrical determinant, a skew determinant and a skew-symmetric determinant.

Show that a skew-symmetrical determinant of odd order has the value zero.

1964

(a) Prove that the reciprocal of any determinant is equal to the $(n-1)$th power of the given determinant.

(b) Prove that a skew-symmetric determinant of even order is a perfect square.

(c) Prove the expansion

$$\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1a_2a_3a_4\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right)$$

1965

(a) Show that a skew symmetric determinant of an odd order vanishes.

(b) Prove that

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

is divisible by λ^3 and find the other factor.

1966

(a) Prove that the reciprocal of a determinant $\triangle$ of the nth order is $\triangle^{n-1}$.

(b) Prove that a skew-symmetric determinant of even order is a perfect square.

(c) Evaluate

$$\begin{vmatrix} 1+a_1 & a_2 & a_3 \ldots\ldots & a_n \\ a_1 & 1+a_2 & a_3 \ldots\ldots & a_n \\ a_1 & a_2 & 1+a_3 \ldots\ldots & a_n \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ a_1 & a_2 & a_3 \ldots\ldots & 1+a_n \end{vmatrix}$$

CALCUTTA UNIVERSITY *Hons.* PAPERS

1961

1. Show that

$$\begin{vmatrix} x & a & a & \ldots & a \\ a & x & a & \ldots & a \\ a & a & x & \ldots & a \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a & a & a & \ldots & x \end{vmatrix} = (x-a)^{n-1}\{x+(n-1)a\},$$

where the determinant is of nth order with diagonal elements all x and other elements all a.

2. Resolve into factors the determinant

$$\begin{vmatrix} (\alpha-\alpha')^2 & (\alpha-\beta')^2 & (\alpha-\gamma')^2 \\ (\beta-\alpha')^2 & (\beta-\beta')^2 & (\beta-\gamma')^2 \\ (\gamma-\alpha')^2 & (\gamma-\beta')^2 & (\gamma-\gamma')^2 \end{vmatrix}.$$

1962

1. Show that the determinant

$$\begin{vmatrix} 0 & (\alpha-\beta)^2 & (\alpha-\gamma)^2 \\ (\beta-\alpha)^2 & 0 & (\beta-\gamma)^2 \\ (\gamma-\alpha)^2 & (\gamma-\beta)^2 & 0 \end{vmatrix}$$ is equal to the product of two determinants

$$\begin{vmatrix} -1 & -2\alpha & \alpha^2 \\ -1 & -2\beta & \beta^2 \\ -1 & -2\gamma & \gamma^2 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}.$$

Hence or otherwise, show that $\Delta=2\,(\alpha-\beta)^2\,(\beta-\gamma)^2\,(\gamma-\alpha)^2$.

1963

Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}=(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)$

Deduce that

$$\begin{vmatrix} \beta^2\gamma^2+\alpha^2\delta^2 & \beta\gamma+\alpha\delta & 1 \\ \gamma^2\alpha^2+\beta^2\delta^2 & \gamma\alpha+\beta\delta & 1 \\ \alpha^2\beta^2+\gamma^2\delta^2 & \alpha\beta+\gamma\delta & 1 \end{vmatrix}=(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta) \times(\alpha-\delta)\,(\beta-\delta)\,(\gamma-\delta).$$

1965

(a) Define the *adjoint* of a determinant and prove that if D is a determinant of order n and D' is its adjoint, then $D'=D^{n-1}$.

(b) Prove that

$$\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix}=x^3(x+a+b+c+d).$$

CALCUTTA UNIVERSITY PASS PAPERS

1965

(a) Show that the value of a determinant of the third order remains unaltered if the rows are changed into columns and the columns into rows.

(b) Show that

$$\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix}=\begin{vmatrix} x & c & b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$$

DEHLI UNIVERSITY Hons. PAPERS

1963

(a) If $\triangle$ is a determinant of third order and $\triangle'$ the reciprocal determinant of Δ, show that $\Delta\Delta'=\Delta^2$.

(b) Show that a skew-symmetric determinant of odd order is zero.

(c) Solve the equations

$x+y+z=1,\ ax+by+cz=k,\ a^2x+b^2y+c^2z=k^2.$

1965

(a) Show that

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}$$

has $a+b\lambda+c\lambda^2+d\lambda^3$ as a factor, where λ is a root of $x^4=1$. Hence or otherwise show that the determinant is equal to

$$(a+b+c+d)\,(a-b+c-d)\quad\{(a-c)^2+(b-d)^2\}.$$

(b) If $\triangle=|a_{ij}|$, $\triangle'=|A_{ij}|$ be two determinants of order n such that A_{ij} is the cofactor of a_{ij} in $\triangle$, prove that $\triangle'=\triangle^{n-1}$. What happens when $\triangle=0$?

DELHI UNIVERSITY PASS PAPERS

1965

(a) Explain the use of determinants in solving linear simultaneous equations.

(b) Calculate the value of the determinant

$$\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix}$$

GAUHATI UNIVERSITY Hons. PAPERS

1963

1. Prove the rule for the multiplication of two determinants of the same order.

If r_r is the sum of the rth powers of the roots of an equation of degree n, then the determinant

$$\begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{vmatrix}$$

is equal to the square of the product of the differences of the roots taken in pairs.

1964

1. Write down as a determinant the product of

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Hence show that the product of two expressions of the form $(p^3+q^3+r^3-3pqr)$ is of the same form.

2. Prove that if

$$\begin{vmatrix} 1+x & 1-x & 1-x & \dots & 1-x \\ 1-x & 1+x & 1-x & \dots & 1-x \\ 1-x & 1-x & 1+x & \dots & 1-x \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1-x & 1-x & 1-x & \dots & 1+x \end{vmatrix}$$

be a determinant of the nth order, then

$$D=(2x)^{n-1}\{n-(n-2)x\}.$$

1965

(a) Prove that the determinant

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

is a perfect square, and find its value.

(b) Prove that if the cofactors of the elements of a certain row (or column) are multiplied, in order, by the elements of another row (or column), show that the sum of the products is zero

(c) Show that a skew-symmetric determinant of odd order vanishes.

RAJASTHAN UNIVERSISY B.Sc. PAPERS

1962

1. Prove that

$$\begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ac \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix} \times \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$
$$=(a^3+b^3+c^3-3abc)^3.$$

1963

1. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$$=(a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a).$$

1965

(b) If ω is the cube root of unity, prove that

$$a+b\omega+c\omega^2$$

is a factor of

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

and hence find the value of the determinant.

1966

(a) Prove that $x=-1$ is a root of the equation

$$\begin{vmatrix} 2-x & 3 & 3 \\ 3 & 4-x & 5 \\ 3 & 5 & 4-x \end{vmatrix}=0$$

and find the other roots.

AGRA UNIVERSITY B.Sc. PAPERS

1960

1. (a) Prove that the product of two determinants each of the third order is a determinant of third order.

(b) Express $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

as a product of two determinants, and find its value.

1961

1. (a) If $a+b+c=0$, solve the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0.$$

(b) Use the properties of *determinants to solve the* equations

$$a_r x+b_r y+c_r z+d_r=0,\ r=1, 2, 3.$$

1962

1. Show that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\ (a+b+c)^3.$$

2. Find the value of $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$.

1963

1. Show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

2. Solve the equation

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$$

1964

1. Find the value of $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$.

2. If

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0, \text{ prove that } xyz=-1$$

1965

1. (a) Evaluate

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

(b) Show that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}=2abc(a+b+c)^3$$

1966

2. (a) Show that

$$\begin{vmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_4 & -B_3 & C_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2,$$

the capital letters denoting the minors of the corresponding small letters in the determinant on the right.

(b) Evaluate

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

VIKRAM UNIVERSITY B. Sc. PAPERS

1962

1. Find the value of

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}.$$

1963

1. Show that

$$\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{c^2+a^2}{b} \end{vmatrix} = 4abc.$$

1961

1. Show that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

1965

2. (a) Prove that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$=(a^3+b^3+c^3-3abc)^2.$$

1966

1. (a) Prove that

$$\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix}$$

$$=-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

SAGAR UNIVERSITY B. Sc. PAPERS

1962

1. Prove that $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}=(x-2y+z)^2.$

1963

1. Calculate the value of the determinant

$$\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

2. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}=2abc\,(a+b+c)^3$$

1964

(a) Prove that if two rows or two columns of a determinant are identical, the determinant vanishes.

(b) Express the following determinant as a product of two determinants and find its value :

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}.$$

1965

5. (a) Explain the use of determinants in solving linear simultaneous equations.

(b) Calculate the value of the determinant :—

$$\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

1966

1. (a) Prove that if two rows or two columns of a determinant are identical, the determinants vanishes.

(b) Evaluate.

$$\begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & a^2 & d & 1 \end{vmatrix}$$

BIHAR UNIVERSITY PAPERS

1965

2. (a) Find the value of

$$\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$$

1966

Prove that

$$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)\,(x-1)^3.$$

BIHAR UNIVERSITY HONS. PAPERS

1966

1. (a) Show that

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix},$$

where $r^2 = x^2+y^2+z^2$ and $u^2 = yz+zx+xy$.

(b) If $A_1, B_1, C_1 \ldots$ are the cofactors of $a_1, b_1, c_1 \ldots$ in the determinant,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2.$$

then show that

KERALA UNIVERSITY 1965
VENKATESWARA AND JIWAJI 1966

Show that $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6.$

1965
MYSORE UNIVERSITY PAPERS

(a) Prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2a & c-a-b \end{vmatrix}$$

is perfect cube.

1966

(a) Show that

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$$\begin{vmatrix} x & l & m & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = (x-a)\,(x-b)\,(x-c)$$

Gorakhpur 67, Meerut 67, Rajasthan 67

$$\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1+t \end{vmatrix} = xyzt\left(1-\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right),$$

(b) Prove that

$$\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & o \end{vmatrix} = (x-2y+z)^2$$

Lucknow 67

2. (a) Prove that

$$U = ax^4+4bx^3+6cx^2+4dx+e$$
$$U_{11} = ax^2+2bx+c$$
$$U_{12} = bx^2+2cx+d$$
$$U_{22} = cx^2+2dx+e,$$

$$\begin{vmatrix} a & b & c & U_{11} \\ b & c & d & U_{12} \\ c & d & e & U_{22} \\ U_{11} & U_{12} & U_{22} & 0 \end{vmatrix} = -U \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

Calcutta 67

4. (a) Factorise $\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha^2, & \beta^2, & \gamma^2 \\ \beta+\gamma, & \gamma+\alpha, & \alpha+\beta \end{vmatrix}$ into four linear factors.

Mysore 67

5 (a) Show that a determinant is unaltered by changing its rows into columns and its columns into rows.

(b) Show that :—

$$\begin{vmatrix} a+b & b+c & c+a \\ b'+c' & c'+a' & a'+b' \\ c''+a'' & a''+b'' & b''+c'' \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b' & c' & a' \\ c'' & a'' & b'' \end{vmatrix}$$

(c) Evaluate :—

$$\begin{vmatrix} 3 & 7 & 9 & 6 \\ 8 & 4 & 5 & 8 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$$

Allahabad 67. 7. (a) Prove that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}=4abc$$

(b) Solve the following by Cramer's rule !

$$x+y+z=11,$$
$$2x-6y-z=0,$$
$$3x+4y+2z=0.$$

) Solve the equation

$$\begin{vmatrix} & x \\ & x^2 \\ & x^3 \\ c & x^4 \end{vmatrix}=0.$$

(b) Show that equations

$a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$

possess a common root only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^2$$

Agra 68. 4. Prove that

$$\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}=8abc.$$